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EQUATION AND
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PROBLEMS**

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Abstract

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ON THE ESTIMATION OF THE NON-OSCILLATION INTERVAL OF A DIFFERENCE EQUATION AND DIFFERENCE BOUNDARY VALUE PROBLEMS

A. L. TEPTIN

In the present paper we consider the difference equation

$$L_h[u_x^h] \equiv \frac{\Delta^n u_x^h}{h^n} - \sum_{k=0}^{n-1} p_{xk}^h \frac{\Delta^k u_x^h}{h^k} = 0, \quad (1)$$

where $h = \frac{b-a}{M}$ ($M \geq 2n$ is an integer); $u_x^h = u_h(t_x)$, $t_x = a + hx$, $x = 0, 1, \dots, M$; $\Delta^0 u_x^h = u_x^h$, $\Delta^1 u_x^h = \Delta u_x^h = u_{x+1}^h - u_x^h$, $\Delta^k u_x^h = \Delta(\Delta^{k-1} u_x^h)$ ($k = 1, 2, \dots, n$). The coefficients p_{xk}^h ($k = 0, 1, \dots, n-1$) are assumed to be defined on the set g_{M-n} : $x = 0, 1, \dots, M-n$, for any $h = \frac{b-a}{M}$. A solution u_x^h of equation (1) will be considered on the set g_M :

$$x = 0, 1, \dots, M \quad \text{for each} \quad h = \frac{b-a}{M}.$$

We shall agree to assign to the value $u_{x_0}^h = 0$ ($x_0 \in g_M$) the sign opposite to the sign of $u_{x_0+1}^h$, if $u_0^h = u_1^h = \dots = u_{x_0}^h = 0$; in all other cases we shall assign to the value $u_{x_0}^h = 0$ the sign opposite to the sign of $u_{x_0-1}^h$. We shall say that u_x^h on the set g_M has a change of sign at the point x^* , if $x^* \leq M-1$ and the signs of $u_{x^*}^h$ and $u_{x^*+1}^h$ are opposite. If, moreover, $u_{x^*}^h \neq 0$ and $u_{x^*+1}^h \neq 0$, then we shall say that u_x^h has a change of sign at the point x^* in the narrow sense.

We shall call equation (1) non-oscillatory on the set g_M for a given $h = \frac{b-a}{M}$, if every nontrivial solution of it for the indicated h has on this set no more than $n-1$ changes of sign. The interval $[a, b]$ will be called an interval of non-oscillation of equation (1) if this equation is non-oscillatory on the set g_M for each $h = \frac{b-a}{M} < H$, where H is some known number.

Questions of non-oscillation in the theory of difference equations play the same important role as the analogous questions in the theory of differential equations (see [1, 2]).

Below we give an estimate of a function defined on the set g_M and having on it n changes of sign, analogous to the well-known estimate of G. A. Bessmertnykh and A. Yu. Levin for an n -times differentiable function [3].

On the basis of this estimate, a criterion for the non-oscillation of equation (1) for fixed h is obtained, as well as an estimate of the length of the non-oscillation interval of this equation. Next, an existence and uniqueness theorem for the solution is proved, and theorems on difference inequalities for a nonlinear difference boundary value problem are proved, and a stability condition for this problem is obtained.

§ 1. Let E_m be an m -dimensional vector space, $\bar{X} \subset E_m$ a closed convex bounded set. A point $u \in \bar{X}$ is called an extreme point of the set \bar{X} if it is not an interior point of any segment belonging to \bar{X} .

A functional $f(u)$, defined on E_m , is called convex if

$$f\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{2}[f(u_1) + f(u_2)]$$

for any $u_1, u_2 \in E_m$.

The results of this section are based essentially on the following assertion [4], which we formulate as a lemma.

Lemma 1. *Let $X \subset E_m$ be a closed bounded set, \bar{X} its convex closure, $X_0 \subset \bar{X}$.*

If the difference $X \setminus X_0$ contains no extreme points of the set \bar{X} , then

$$\max_{u \in X} f(u) = \max_{u \in X_0} f(u)$$

for any continuous convex functional $f(u)$.

This assertion follows from the Krein–Milman theorem on extreme points [5].

For fixed

$$h = \frac{b - a}{M}$$

consider the space E_{M+1} of functions u_x^h defined on the set g_M , with norm

$$\|u_x^h\| = \max_{x \in g_M} |u_x^h|.$$

The function u_x^h may be regarded as the vector

$$u = \{u_0^h, u_1^h, \dots, u_M^h\}.$$

Thus, E_{M+1} may be regarded as an $(M + 1)$ -dimensional vector space.

Denote by Q the set of functions $u_x^h \in E_{M+1}$ satisfying the condition

$$\frac{\Delta^n u_x^h}{h^n} = f_x^h (x \in g_{M-n}), \quad \max_{x \in g_{M-n}} |f_x^h| = A, \quad (2)$$

where f_x^h is a given function, and having on g_M at least n changes of sign, including zero or a change of sign at the points $x = 0$ and $x = M - 1$.

Lemma 2. *The set Q is bounded and closed.*

Proof. By virtue of Corollary 2 of Lemma 4 of [1], $\frac{\Delta^k u_x^h}{h^k}$ for any $u_x^h \in Q$ has on the set g_{M-k} , $x = 0, 1, \dots, M - k$, no less...

of it $n - k$ changes of sign ($k = 0, 1, \dots, n - 1$). Therefore, for each $x \in g_{M-k}$ one can indicate the nearest point x_0 to it at which

$$\frac{\Delta^k u_x^h}{h^k}$$

has a change of sign. If $x_0 < x$, then

$$\begin{aligned} \left| \frac{\Delta^k u_x^h}{h^k} \right| &\leq \left| \frac{\Delta^k u_x^h}{h^k} - \frac{\Delta^k u_{x_0}^h}{h^k} \right| = \left| \sum_{s=x_0}^{x-1} \frac{\Delta^{k+1} u_s^h}{h^{k+1}} h \right| \leq \\ &\leq \sum_{s=0}^{M-k-1} \left| \frac{\Delta^{k+1} u_s^h}{h^{k+1}} \right| h \leq Mh \max_{x \in g_{M-k-1}} \left| \frac{\Delta^{k+1} u_x^h}{h^{k+1}} \right| \quad (k = 0, 1, \dots, n - 1). \end{aligned}$$

Analogous inequalities are obtained also when $x_0 > x$. From these inequalities, in view of (2), it follows that

$$\|u_x^h\| \leq AM^n h^n = A(b - a)^n$$

for any $u_x^h \in Q$.

The boundedness of Q has been proved. Let us prove its closedness. Consider a sequence $\{u_{x_i}^h\} \subset Q$. Let

$$\lim_{i \rightarrow \infty} u_{x_i}^h = u_x^{*h}. \quad (3)$$

Then, by virtue of the equality

$$\Delta^n u_x^h = \sum_{k=0}^n (-1)^{n-k} C_n^k u_{x+k}^h$$

[6], we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{\Delta^n u_{xi}^h}{h^n} &= \frac{1}{h^n} \lim_{i \rightarrow \infty} \sum_{k=0}^n (-1)^{n-k} C_n^k u_{x+k,i}^h = \\ &= \frac{1}{h^n} \sum_{k=0}^n (-1)^{n-k} C_n^k u_{x+k}^{*h} = \frac{\Delta^n u_x^{*h}}{h^n}, \end{aligned}$$

whence

$$\frac{\Delta^n u_x^{*h}}{h^n} = \lim_{i \rightarrow \infty} f_x^h = f_x^h \quad (x \in g_{M-n}),$$

i.e. u_x^{*h} satisfies condition (2).

Suppose that u_x^{*h} has fewer than n changes of sign on g_M . But then, by virtue of (3), for sufficiently large i , u_{xi}^h must have fewer than n changes of sign on g_M , which contradicts the definition of the set Q . Hence u_x^{*h} has at least n changes of sign on g_M .

Finally, if we assume that

$$u_0^{*h} u_1^{*h} > 0 \quad \text{or} \quad u_{M-1}^{*h} u_M^{*h} > 0, \quad (4)$$

then, by virtue of (3), for sufficiently large i

$$u_{0i}^h u_{1i}^h > 0 \quad \text{or} \quad u_{M-1,i}^h u_{Mi}^h > 0,$$

i.e. u_{xi}^h has neither a zero nor a change of sign at the point $x = 0$ or $x = M - 1$, which also contradicts the definition of the set Q . Hence the relation

$$(4) \text{ is impossible, i.e., } u_x^{*h} \text{ has a zero or a change of sign at the points } x = 0 \text{ and } x = M - 1.$$

Thus, $u_x^{*h} \in Q$. Hence the set Q is closed. The lemma is proved.

Consider the subset $Q_0 \subset Q$, consisting of functions that have no change of sign on g_M in the strict sense and satisfy one of the following conditions:

$$\text{a) } u_0^h = \dots = u_{i-1}^h = 0, \quad u_{M-n+i+1}^h = \dots = u_M^h = 0 \quad (1 \leq i \leq n-1); \quad (5)$$

$$\text{b) } u_0^h = 0, \quad u_{M-n}^h = \dots = u_{M-1}^h = 0, \quad u_M^h \neq 0; \quad (6)$$

$$\text{c) } u_0^h = 0, \quad u_{M-n+1}^h = \dots = u_{M-1}^h = 0, \quad \text{sign}(u_{M-n}^h u_M^h) = (-1)^n; \quad (7)$$

$$\text{d) } u_0^h \neq 0, \quad u_1^h = \dots = u_n^h = 0, \quad u_M^h = 0; \quad (8)$$

$$\text{e) } u_1^h = \dots = u_{n-1}^h = 0, \quad \text{sign}(u_0^h u_n^h) = (-1)^n, \quad u_M^h = 0; \quad (9)$$

$$\text{f) } u_1^h = \dots = u_n^h = 0 \quad \text{or} \quad u_1^h = \dots = u_{n-1}^h = 0,$$

$$\text{sign}(u_0^h u_n^h) = (-1)^n \quad \text{and} \quad u_{M-n}^h = \dots = u_{M-1}^h = 0 \quad \text{or} \quad u_{M-n+1}^h = \dots$$

$$\dots = u_{M-1}^h = 0, \quad \text{sign}(u_{M-n}^h u_M^h) = (-1)^n; \quad (10)$$

$$\text{g) } u_0^h = u_M^h = 0, \quad u_1^h \neq 0, \quad u_{M-1}^h \neq 0, \quad u_{x_0}^h = \dots = u_{x_0+n-2}^h = 0 \quad (11)$$

$$(1 < x_0 < M - n + 1);$$

$$\text{h) } u_0^h = u_M^h = 0, \quad u_1^h \neq 0, \quad u_{M-1}^h \neq 0, \quad u_{x_0}^h = \dots = u_{x_0+n-3}^h = 0,$$

$$\text{sign}(u_{x_0-1}^h u_{x_0+n-2}^h) = (-1)^{n-1} \quad (1 < x_0 < M - n + 2). \quad (12)$$

Let \bar{Q} be the convex closure of the set Q .

Lemma 3. The difference $Q \setminus Q_0$ contains no extreme points of the set \bar{Q} .

Proof. It is necessary to prove that any function $u_x^h \in Q \setminus Q_0$ is not an extreme point of the set \bar{Q} . Here the following cases are possible:

- 1) u_x^h has on g_M a change of sign in the strict sense; in all the following cases it is assumed that u_x^h has no changes of sign in the strict sense;
- 2) u_x^h has on g_M fewer than n zeros; in cases 3)–7) we shall assume that u_x^h has on g_M at least n zeros;
- 3) $u_0^h \neq 0$, $u_1^h = \dots = u_m^h = 0$, $u_{m+1}^h \neq 0$, where either $m < n - 1$, or $m = n - 1$, $\text{sign}(u_0^h u_{m+1}^h) = (-1)^m$;
- 4) $u_M^h \neq 0$, $u_{M-m}^h = \dots = u_{M-1}^h = 0$, $u_{M-m-1}^h \neq 0$, where either $m < n - 1$, or $m = n - 1$, $\text{sign}(u_{M-m-1}^h u_M^h) = (-1)^m$;
- 5) $u_0^h = \dots = u_{i-1}^h = u_{M-m+1}^h = \dots = u_M^h = 0$, $u_i^h \neq 0$, $u_{M-m}^h \neq 0$, $i > 1$ or $m > 1$, $i + m < n$;

- 6) $u_0^h = u_M^h = 0$, $u_1^h \neq 0$, $u_{M-1}^h \neq 0$, and the remaining zeros are not all consecutive;
- 7) $u_0^h = u_M^h = 0$, $u_1^h \neq 0$, $u_{M-1}^h \neq 0$, $u_{x_0}^h = \dots = u_{x_0+n-3}^h = 0$ ($1 < x_0 < M - n + 2$), $\text{sign}(u_{x_0-1}^h u_{x_0+n-2}^h) = (-1)^{n-2}$.

Construct functions

$$z_{x_1}^h = u_x^h + \varepsilon v_x^h, \quad z_{x_2}^h = u_x^h - \varepsilon v_x^h,$$

where $\varepsilon > 0$, and the function v_x^h is defined as follows. In cases 1) and 2), when u_x^h has $k \leq n - 1$ zeros on g_M , v_x^h is a solution of the equation

$$\frac{\Delta^k v_x^h}{h^k} = 1, \quad (13)$$

having zeros at the same points as u_x^h . In the remaining cases v_x^h is a solution of the equation

$$\frac{\Delta^{n-1} v_x^h}{h^{n-1}} = 1, \quad (14)$$

having on the set g_M $n - 1$ zeros coinciding with the zeros of u_x^h . These zeros are chosen in accordance with the following rules, each of which corresponds to one of the cases listed above.

- 1) If u_x^h has more than $n - 1$ zeros on g_M , then $n - 1$ of them are chosen arbitrarily, except for the case when $u_0^h = 0$ or $u_1^h = 0$ ($u_M^h = 0$ or $u_{M-1}^h = 0$). In this case the point $x = 0$ or $x = 1$ ($x = M$ or $x = M - 1$) is included among the chosen ones.
- 2) If $\text{sign}(u_0^h u_{m+1}^h) = (-1)^m$ ($m \leq n - 1$), then the points $x = 1, \dots, m - 1$ and $x = M$ or $x = M - 1$ are included among the chosen zeros of the function u_x^h . If, however, $\text{sign}(u_0^h u_{m+1}^h) = (-1)^{m+1}$ ($m < n - 1$), then $x = m$ is added to the points just listed.
- 3) The $n - 1$ chosen zeros of the function u_x^h include the points $x = M - m + 1, \dots, M - 1$ and $x = 0$ or $x = 1$, and in the case

$$\text{sign}(u_{M-m-1}^h u_M^h) = (-1)^{m+1} \quad (m < n - 1)$$

the point $x = M - m$ is added to these points.

- 4) In this case u_x^h has at least $n - i - m$ zeros on the set $x = i + 1, \dots, M - m - 1$. Let x_0 be such a point of this set that

$$u_{x_0}^h = \dots = u_{x_0+k-1}^h = 0 \quad (i + 1 \leq x_0 \leq M - m - k, \quad k \geq 1),$$

$$u_{x_0-1}^h \neq 0, \quad u_{x_0+k}^h \neq 0.$$

Suppose further that

$$\text{sign}(u_{x_0-1}^h u_{x_0+k}^h) = (-1)^{k+1}.$$

If $k \leq n - i - m$, then the $n - 1$ chosen zeros of the function u_x^h include the points

$$x = 0, 1, \dots, i - 2, x_0, x_0 + 1, \dots, x_0 + k - 1,$$

$$M - m + 1, \dots, M \quad \text{for } i > 1$$

or

$$x = 0, 1, \dots, i - 1, x_0, x_0 + 1, \dots, x_0 + k - 1,$$

$$M - m + 2, \dots, M \quad \text{for } m > 1.$$

If $k - (n - i - m) > 0$ is even, then the following $n - 1$ zeros of the function u_x^h are chosen:

$$x = 0, 1, \dots, i - 2, x_0, x_0 + 1, \dots, x_0 + n - i - m - 1,$$

$$M - m + 1, \dots, M \quad \text{for } i > 1 \tag{15}$$

or

$$\begin{aligned} x = 0, 1, \dots, i - 1, x_0, x_0 + 1, \dots, x_0 + n - i - m - 1, \\ M - m + 2, \dots, M \quad \text{for } m > 1. \end{aligned} \tag{16}$$

If $k - (n - i - m) > 0$ is odd, then the zeros of u_x^h are chosen at the points

$$\begin{aligned} x = 0, 1, \dots, i - 1, x_0, x_0 + 1, \dots, x_0 + n - i - m - 2, \\ M - m + 1, \dots, M. \end{aligned} \tag{17}$$

Let now $\text{sign}(u_{x_0-1}^h u_{x_0+k}^h) = (-1)^k$. If $k \leq n - i - m$, then among the chosen $n - 1$ zeros of u_x^h are included the points $x = 0, 1, \dots, i - 1, x_0, x_0 + 1, \dots, x_0 +$

$k - 2, M - m + 1, \dots, M$. If $k - (n - i - m) > 0$ is even, then $n - 1$ zeros of u_x^h are taken at the points (17). If $k - (n - i - m) > 0$ is odd, then the zeros of u_x^h are chosen at the points (15) or (16).

- 6) In this case u_x^h has at least $n - 2$ zeros on the set $x = 2, 3, \dots, M - 2$, and not all of them are consecutive.

Let $x_0 \in g_M$ be such a point that $u_{x_0}^h = \dots = u_{x_0+k-1}^h = 0$ ($2 \leq x_0 \leq M - k - 1, k > 1$), $u_{x_0-1}^h \neq 0, u_{x_0+k}^h \neq 0$. Here the zeros of u_x^h are chosen at the points $x = 0, x_0, x_0 + 1, \dots, x_0 + p - 1, M$ and $n - p - 3$ arbitrary zeros on the set $x = 2, 3, \dots, x_0 - 2, x_0 + k + 1, \dots, M - 2$, where p is the number defined as follows.

If $\text{sign}(u_{x_0-1}^h u_{x_0+k}^h) = (-1)^{k+1}$, then

$$p = \begin{cases} k, & \text{for } k \leq n - 3, \\ n - 3, & \text{for } k - (n - 3) > 0 \text{ even,} \\ n - 4, & \text{for } k - (n - 3) > 0 \text{ odd.} \end{cases}$$

If, however, $\text{sign}(u_{x_0-1}^h u_{x_0+k}^h) = (-1)^k$, then

$$p = \begin{cases} k - 1, & \text{for } k \leq n - 2, \\ n - 3, & \text{for } k - (n - 2) > 0 \text{ even,} \\ n - 4, & \text{for } k - (n - 2) > 0 \text{ odd.} \end{cases}$$

- 7) The zeros of u_x^h are chosen at the points $x = 0, x_0, x_0 + 1, \dots, x_0 + n - 4, M$.

By virtue of the results of [1], the equation

$$\frac{\Delta^{k_0} u_x^h}{h^k} = 0$$

is non-oscillatory on the whole axis for any k . Therefore the solution of equation (14) ((13)) having $n - 1$ (k) zeros on the set g_M exists and is unique.

In each of the cases 1)–7) it is easy to show that, for sufficiently small ε , the functions $z_{x_j}^h$ ($j = 1, 2$) each have at least n sign changes on the set g_M , including a zero or a sign change at the points $x = 0$ and $x = M - 1$. It is also obvious that $z_{x_j}^h$ ($j = 1, 2$) satisfy condition (2). Thus, $z_{x_j}^h \in Q$ ($j = 1, 2$). Finally,

$$\frac{z_{x_1}^h + z_{x_2}^h}{2} = u_x^h.$$

This means that any function $u_x^h \in Q \setminus Q_0$ is an interior point of some interval belonging to \overline{Q} , i.e., u_x^h is not an extreme point of \overline{Q} . The lemma is proved.

Lemma 4.

$$\max_{u_x^h \in Q} \left\{ \max_{x \in g_{M-k}} \left| \frac{\Delta^k u_x^h}{h^k} \right| \right\} = \max_{u_x^h \in Q_0} \left\{ \max_{x \in g_{M-k}} \left| \frac{\Delta^k u_x^h}{h^k} \right| \right\}$$

$$(k = 0, 1, \dots, n-1).$$

Proof. It is easy to see that

$$f(u_x^h) = \max_{x \in g} \left| \frac{\Delta^k u_x^h}{h^k} \right|,$$

where $g \subset g_{M-k}$ is any subset, is a continuous convex functional defined on E_{M+1} . Then the assertion of the present lemma follows immediately from Lemmas 1-3.

Let $Q_1 \subset Q$ be the set consisting of functions satisfying one of the following conditions:

$$u_0^h = \dots = u_{n-2}^h = u_M^h = 0, \quad (18)$$

$$u_1^h = \dots = u_{n-1}^h = u_M^h = 0, \quad (19)$$

$$u_1^h = \dots = u_{n-2}^h = u_M^h = 0, \quad \text{sign}(u_0^h u_{n-1}^h) = (-1)^{n-1}; \quad (20)$$

$Q_2 \subset Q_1$ is a subset consisting of only two functions, one of which satisfies condition (18), the other condition (19); Q_1 is the convex closure of Q_2 .

Lemma 5.

$$\max_{u_x^h \in Q_1} \left\{ \max_{x \in g} \left| \frac{\Delta^k u_x^h}{h^k} \right| \right\} = \max_{u_x^h \in Q_2} \left\{ \max_{x \in g} \left| \frac{\Delta^k u_x^h}{h^k} \right| \right\} \quad (k = 0, 1, \dots, n-1),$$

where $g \subset g_{M-k}$ is any subset.

Proof. Repeating the reasoning given in the proof of Lemma 2, we see that the set Q_1 is bounded and closed. Further, analogously to how this was done in the proof of Lemma 3 (case 2), one can show that the difference $Q_1 \setminus Q_2$ contains no extreme points of the set $\overline{Q_1}$. To complete the proof it remains to refer to Lemma 1.

Denote by $x^{(k)}$ the generalized power [6], i.e.,

$$x^{(k)} = x(x-1) \dots (x-k+1).$$

Lemma 6. If the function $u_x^h \in Q_0$ satisfies conditions (18) or

$$u_0^h = u_{M-n+2}^h = \dots = u_M^h = 0, \quad (21)$$

then

$$\left| \frac{\Delta^k u_x^h}{h^k} \right| \leq \frac{Ak(M-k)^{n-k} h^{n-k}}{n(n-k)!} \quad (x \in g_{M-k}; \quad k = 1, 2, \dots, n-1), \quad (22)$$

where equality is possible only when

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A$$

and $x = 0$ or $x = M - k$.

Proof. Let u_x^h satisfy conditions (18). u_x^h may be regarded as the solution of the boundary-value problem

$$\frac{\Delta^n u_x^h}{h^n} = f_x^h, \quad u_0^h = \dots = u_{n-2}^h = u_M^h = 0. \quad (23)$$

By virtue of the results of [1], the solution of problem (23) exists and is unique for any f_x^h . Hence there exists the Green's function of this problem [7], i.e. a function $G_{x,s}^h$, which, for any fixed $s \in g_{M-n}$, satisfies with respect to x the boundary conditions (18) and the equation

$$\frac{\Delta^n G_{x,s}^h}{h^n} = \frac{\delta_{x,s}}{h} \quad (\delta_{x,s} = 0 \text{ for } x \neq s, \delta_{s,s} = 1).$$

And then

$$u_x^h = \sum_{s=0}^{M-n} G_{x,s}^h f_s^h h \quad [7],$$

whence

$$\frac{\Delta^k u_x^h}{h^k} = \sum_{s=0}^{M-n} \frac{\Delta^k G_{x,s}^h}{h^k} f_s^h h \quad (k = 1, \dots, n-1), \quad (24)$$

where

$$\Delta_x^1 G_{x,s}^h = \Delta_x G_{x,s}^h = G_{x+1,s}^h - G_{x,s}^h, \quad \Delta_x^k G_{x,s}^h = \Delta_x (\Delta_x^{k-1} G_{x,s}^h).$$

For the Green's function of problem (23), one easily finds the analytic expression

$$G_{x,s}^h = \begin{cases} \frac{(x-s-1)^{(n-1)}M^{(n-1)} - (M-s-1)^{(n-1)}x^{(n-1)}}{(n-1)!M^{(n-1)}} h^{n-1}, & \text{for } x \geq s+1, \\ -\frac{(M-s-1)^{(n-1)}x^{(n-1)}}{(n-1)!M^{(n-1)}} h^{n-1}, & \text{for } x \leq s, \end{cases} \quad (25)$$

with the aid of which, without particular difficulty, one can verify that

$$\left| \frac{\Delta_x^k G_{x,s}^h}{h^k} \right| < \frac{\Delta_x^k G_{M-k,s}^h}{h^k} \quad (k = 1, \dots, n-2) \quad (26)$$

for any $x \in g_{M-k}$, $x \neq M-k$, $s \in g_{M-n}$.

From equality (24), by virtue of (2), (25), (26), and the summation formulas for generalized powers [6], we obtain estimate (22); moreover it is obvious that equality in it is possible only in the case $f_x^h = \pm A$ for $x = M-k$, and for $k = n-1$ also for $x = 0$.

If u_x^h satisfies conditions (21), then to prove the lemma it is enough to introduce the new variable $x' = M-x$ and use the result established above.

Remark. By means of the change of variable $x' = x-r$, it is easy to see that Lemma 6 remains valid also on the set $x = r, r+1, \dots, M$ ($r < M$ any integer), if in the statement of the lemma one replaces $(M-k)^{(n-k)}$ by $(M-k-r)^{(n-k)}$ and $x=0$ by $x=r$.

Lemma 7. If the function $u_x^h \in Q$ satisfies conditions (19) or (20) (the conditions

$$u_0^h = u_{M-n+1}^h = \dots = u_{M-1}^h = 0 \quad \text{or} \quad u_0^h = u_{M-n+2}^h = \dots = u_{M-1}^h = 0, \quad (27)$$

$$\text{sign}(u_{M-n+1}^h u_M^h) = (-1)^{n-1},$$

then

$$\left| \frac{\Delta_x^k u_x^h}{h^k} \right| \leq \frac{Ak(M-k)^{(n-k)}h^{n-k}}{n(n-k)!} \quad (k = 1, \dots, n-1) \quad (28)$$

for $x = 1, 2, \dots, M-k$ ($x = 0, 1, \dots, M-k-1$),

$$\left| \frac{\Delta^{n-1} u_0^h}{h^{n-1}} \right| \leq \frac{A(n-1)(M-n+2)h}{n} \quad (29)$$

$$\left(\left| \frac{\Delta^{n-1} u_{M-n+1}^h}{h^{n-1}} \right| \leq \frac{A(n-1)(M-n+2)h}{n} \right).$$

Equalities here are possible only in the case

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A,$$

and in (28) only for $x = 0$ and $x = M - k$, while in (29) only for $n = 2$.

Proof. If u_x^h satisfies conditions (19), then, applying Lemma 6 on the set $x = 1, 2, \dots, M$, we obtain

$$\left| \frac{\Delta^k u_x^h}{h^k} \right| \leq \frac{Ak(M-k-1)^{(n-k)} h^{n-k}}{n(n-k)!} \quad (30)$$

$$(x = 1, 2, \dots, M - k; k = 1, \dots, n - 1),$$

and equality is possible only when

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A.$$

At the point $x = 0$,

$$\frac{\Delta^{n-1} u_0^h}{h^{n-1}} = \frac{\Delta^{n-1} u_1^h}{h^{n-1}} - \frac{\Delta^n u_0^h}{h^n} h,$$

whence, by virtue of (2) and (30),

$$\begin{aligned} \left| \frac{\Delta^{n-1} u_0^h}{h^{n-1}} \right| &\leq \left| \frac{\Delta^{n-1} u_1^h}{h^{n-1}} \right| + \left| \frac{\Delta^n u_0^h}{h^n} \right| h \leq \\ &\leq \frac{A(n-1)(M-n)h}{n} + Ah \leq \frac{A(n-1)(M-n+2)h}{n}. \end{aligned}$$

It is clear that equality here is possible only when

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A,$$

$n = 2$.

If u_x^h satisfies conditions (20), then the validity of the present lemma follows, by virtue of Lemma 5, from the preceding part of the proof and Lemma 6.

Finally, if u_x^h satisfies conditions (27), then to prove the lemma it suffices to introduce the new variable $x' = M - x$ and use the preceding results.

Remark. Obviously, Lemma 7, with the corresponding corrections, is also valid on the set $x = r, r + 1, \dots, M$ ($r < M$ is any integer).

Lemma 8. If the function $u_x^h \in Q_0$ satisfies one of the conditions (6)–(12), then

$$\left| \frac{\Delta^k u_x^h}{h^k} \right| < \frac{Ak(M-k)^{(n-k)}h^{n-k}}{n(n-k)!} \quad (x \in g_{M-k}, k = 1, 2, \dots, n-1).$$

Proof. Consider the case when u_x^h satisfies condition (8) or (9). In this case Lemma 6 is applicable to u_x^h on the set $x = 1, 2, \dots, M$, by virtue of which the estimate (30) is valid.

From (8) and (9) it follows that

$$\frac{\Delta^k u_1^h}{h^k} = 0 \quad (k = 0, 1, \dots, n-2), \quad \frac{\Delta^{n-1} u_0^h}{h^{n-1}} - \frac{\Delta^{n-1} u_1^h}{h^{n-1}} \leq 0.$$

Therefore

$$\left| \frac{\Delta^k u_0^h}{h^k} \right| = \left| \frac{\Delta^k u_1^h}{h^k} - \frac{\Delta^k u_0^h}{h^k} \right| = \left| \frac{\Delta^{k+1} u_0^h}{h^{k+1}} \right| h \quad (k = 0, 1, \dots, n-2),$$

$$\left| \frac{\Delta^{n-1} u_0^h}{h^{n-1}} \right| \leq \left| \frac{\Delta^{n-1} u_1^h}{h^{n-1}} - \frac{\Delta^{n-1} u_0^h}{h^{n-1}} \right| = \left| \frac{\Delta^n u_0^h}{h^n} \right| h.$$

Hence, by virtue of (2),

$$\left| \frac{\Delta^k u_0^h}{h^k} \right| \leq Ah^{n-k} \quad (k = 0, 1, \dots, n-1). \quad (31)$$

But for $M \geq 2n$,

$$\frac{k(M-k-1)^{(n-k)}}{n(n-k)!} \geq 1.$$

Therefore from (31) we obtain

$$\left| \frac{\Delta^k u_0^h}{h^k} \right| \leq \frac{Ak(M-k-1)^{(n-k)}h^{n-k}}{n(n-k)!} \quad (k = 1, \dots, n-1),$$

and (30) together with the last inequality proves the lemma in the case (8) or (9). In the remaining cases the proof is carried out in an analogous manner, and in case (12) Lemma 7 is used.

Lemma 9. For any function $u_x^h \in Q_0$ the estimate

$$|u_x^h| \leq \frac{A(n-1)^{n-1}M^n h^n}{n^n n!} \quad (x \in g_M),$$

is valid, with equality possible only when

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A, \quad x = \frac{M}{2}, \quad n = 2.$$

Proof. u_x^h may be regarded as the solution of the n -point boundary-value problem for the equation

$$\frac{\Delta^n u_x^h}{h^n} = f_x^h$$

with zero boundary conditions. By virtue of the results of [1], for the problem mentioned there exists a Green's function $G_{x,s}^h$, and therefore

$$u_x^h = \sum_{s=0}^{M-n} G_{x,s}^h f_s^h h.$$

It also follows from the results of [1] that, for each $x \in g_M$, $G_{x,s}^h$, as a function of s , does not change sign on the set g_{M-n} . Hence, taking into account the inequality $|f_s^h| \leq A$ ($x \in g_{M-n}$), we obtain

$$|u_x^h| \leq \sum_{s=0}^{M-n} |G_{x,s}^h| \cdot |f_s^h| h \leq \sum_{s=0}^{M-n} |G_{x,s}^h| Ah = \left| \sum_{s=0}^{M-n} G_{x,s}^h Ah \right|. \quad (32)$$

But

$$v_x^h = \sum_{s=0}^{M-n} G_{x,s}^h Ah \quad (33)$$

is a polynomial of degree n , whose zeros coincide with the zeros of u_x^h . If $u_0^h = u_M^h = 0$, then

$$v_x^h = P(t_x), \quad (34)$$

where

$$P(t) = \frac{A(t-a)(t-\alpha_1)\cdots(t-\alpha_{n-2})(t-b)}{n!},$$

$$\alpha_i = a + hx_i \quad (i = 1, \dots, n-2),$$

and x_1, \dots, x_{n-2} are the zeros of u_x^h between the points $x = 0$ and $x = M$. For the polynomial $P(t)$ the following estimate is known (see, for example, [3]):

$$|P(t)| \leq \frac{A(n-1)^{n-1}(b-a)^n}{n^n n!} \quad (t \in [a, b]). \quad (35)$$

Equality here is possible only when all α_i ($i = 1, \dots, n-2$) coincide with the point a or with the point b . But in the present case $a < \alpha_1 < \dots < \alpha_{n-2} < b$, and therefore inequality (35) is strict for all n and t , except for $n = 2$, $t = \frac{a+b}{2}$.

From (32)–(35) and the remark on inequality (35), the validity of the present lemma follows in the case when $u_0^h = u_M^h = 0$, i.e., when u_x^h satisfies one of the conditions (5), (11), (12).

If u_x^h satisfies condition (8) or (9), then the preceding arguments remain valid on the set $x = 1, \dots, M$. Therefore,

$$|u_x^h| \leq \frac{A(n-1)^{n-1}(M-1)^n h^n}{n^n n!} \quad (x = 1, \dots, M). \quad (36)$$

And at the point $x = 0$, by virtue of (31),

$$|u_0^h| \leq Ah^n. \quad (37)$$

Since for $M \geq 2n$

$$\frac{(n-1)^{n-1} M^n}{n^n n!} > 1,$$

then from (37) it follows that

$$|u_0^h| < \frac{A(n-1)^{n-1} M^n h^n}{n^n n!}.$$

This inequality, together with (36), proves the lemma in cases (8) and (9). The proof is carried out analogously also when u_x^h satisfies one of the conditions (6), (7), (10).

The lemma is proved.

We now give the main theorems of this paragraph.

Theorem 1. If the function $u_x^h \in E_{M+1}$ has at least n changes of sign on the set g_M , including zero or a change of sign at the points $x = 0$ and $x = M - 1$, then

$$|u_x^h| \leq \frac{A(n-1)^{n-1}M^n h^n}{n^n n!} \left(x \in g_M, \quad A = \max_{x \in g_{M-n}} \left| \frac{\Delta^n u_x^h}{h^n} \right| \right),$$

and equality is possible only when

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A, \quad n = 2, \quad x = \frac{M}{2}.$$

Proof. Denote

$$\frac{\Delta^n u_x^h}{h^n} = f_x^h.$$

The function u_x^h belongs to the set Q corresponding to this f_x^h . The validity of the theorem now follows directly from Lemmas 4 and 9.

Theorem 2. If the function $u_x^h \in E_{M+1}$ has at least n changes of sign on the set g_M , including zero or a change of sign at the points $x = 0$ and $x = M - 1$, then

$$\left| \frac{\Delta^k u_x^h}{h^k} \right| \leq \frac{Ak(M-k)^{n-k} h^{n-k}}{n(n-k)!}$$

$$\left(x \in g_{M-k}; \quad k = 1, \dots, n-1; \quad A = \max_{x \in g_{M-n}} \left| \frac{\Delta^n u_x^h}{h^n} \right| \right),$$

and equality is possible only when

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A, \quad k = n-1, \quad x = 0 \text{ and } x = M-k.$$

Proof. Again denote

$$\frac{\Delta^n u_x^h}{h^n} = f_x^h.$$

The function u_x^h belongs to the set Q corresponding to this f_x^h . We define the subset $Q_0 \subset Q$ as before.

Let $n = 2$. Then Lemma 6 or 8 is applicable to any function $y_x^h \in Q_0$. By the cited lemma,

$$\left| \frac{\Delta y_x^h}{h} \right| \leq \frac{A(M-1)h}{2} \quad (x \in g_{M-1}),$$

and equality is possible only in the case

$$\frac{\Delta^2 y_x^h}{h^2} \equiv \pm A, \quad x = 0 \text{ or } x =$$

$= M - 1$. Hence, by Lemma 4, the assertion of the present theorem follows for $n = 2$.

Suppose that the theorem is true for some $n = m \geq 2$, and consider the case $n = m + 1$.

Let $y_x^h \in Q_0$. If y_x^h satisfies condition (5) for $i = 1$ or $i = n - 1$, or one of conditions (6)–(12), then, by Lemmas 6 and 8, estimate (22) is valid for y_x^h , equality being possible only when

$$\frac{\Delta^n y_x^h}{h^n} \equiv \pm A, \quad x = 0 \text{ or } x = M - k.$$

Since, for $k < n - 1$,

$$(M - k)^{(n-k)} < (M - k)^{n-k},$$

it follows from (22) that

$$\left| \frac{\Delta^k y_x^h}{h^k} \right| \leq \frac{Ak(M - k)^{n-k} h^{n-k}}{n(n - k)!} \quad (x \in g_{M-k}; k = 1, 2, \dots, n - 1), \quad (38)$$

where equality is possible only when

$$\frac{\Delta^n y_x^h}{h^n} \equiv \pm A, \quad x = 0 \text{ or } x = M - k, \quad k = n - 1.$$

If, however, $y_x^h \in Q_0$ satisfies condition (5) for $2 \leq i \leq n - 2 = m - 1$, then consider the function

$$v_x^h = \frac{\Delta y_x^h}{h}.$$

It is easy to see that

$$v_0^h = v_{M-1}^h = 0.$$

By Corollary 1 of Lemma 4 of [1], v_x^h has at least $n - 1 = m$ changes of sign on the set g_{M-1} . Finally,

$$\max_{x \in g_{M-m-1}} \left| \frac{\Delta^m v_x^h}{h^m} \right| = \max_{x \in g_{M-n}} \left| \frac{\Delta^n y_x^h}{h^n} \right| = A.$$

Thus v_x^h on the set g_{M-1} satisfies the conditions of the present theorem for $n = m$. And since for $n = m$ the theorem is assumed to be true, we have

$$\left| \frac{\Delta^{k-1} v_x^h}{h^{k-1}} \right| \leq \frac{A(k-1)(M-k)^{m-k+1} h^{m-k+1}}{m(m-k+1)!} \quad (x \in g_{M-k}; k = 2, \dots, m).$$

Hence, for $n = m + 1$,

$$\left| \frac{\Delta^k y_x^h}{h^k} \right| \leq \frac{A(k-1)(M-k)^{n-k} h^{n-k}}{(n-1)(n-k)!} \quad (x \in g_{M-k}; k = 2, \dots, n-1).$$

Since

$$\frac{k-1}{n-1} < \frac{k}{n}$$

for all $0 < k < n$, from the last inequality we obtain

$$\left| \frac{\Delta^k y_x^h}{h^k} \right| < \frac{Ak(M-k)^{n-k} h^{n-k}}{n(n-k)!} \quad (x \in g_{M-k}; k = 2, \dots, n-1).$$

Finally, by Theorem 1,

$$|v_x^h| \leq \frac{A(m-1)^{m-1}(M-1)^m h^m}{m^m m!} \quad (x \in g_{M-1}).$$

or

$$\left| \frac{\Delta y_x^h}{h} \right| \leq \frac{A(n-2)^{n-2}(M-1)^{n-1} h^{n-1}}{(n-1)^{n-1}(n-1)!} \quad (x \in g_{M-1}).$$

Hence, by virtue of the inequality

$$\frac{(n-2)^{n-2}}{(n-1)^{n-1}} < \frac{1}{n} \quad \text{for } n > 2,$$

it follows that

$$\left| \frac{\Delta y_x^h}{h} \right| < \frac{A(M-1)^{n-1} h^{n-1}}{n(n-1)!} \quad (x \in g_{M-1}).$$

Thus, we have established that for $n = m + 1$ estimate (38) is valid for any function $y_x^h \in Q_0$. But then, by Lemma 4, it is also valid, for this n , for any

function $u_x^h \in Q$. Hence the validity of the theorem for arbitrary n follows by induction.

It is easy to see that Theorems 1 and 2, with the corresponding modifications, are also valid on the set $x = r, r + 1, \dots, M$ for any integer $r \leq M - 2n$.

§ 2. The preceding theorems make it possible to prove easily the following criterion for the non-oscillation of difference equations.

Theorem 3. *If, for a given h , the coefficients of equation (1) satisfy the inequalities*

$$|p_{xk}^h| \leq L_k^h \quad (k = 0, 1, \dots, n - 1; x \in g_{M-n}),$$

$$\sum_{k=1}^{n-1} \frac{L_k^h k (M - k)^{n-k} h^{n-k}}{n(n-k)!} + \frac{L_0^h (n-1)^{n-1} M^n h^n}{n^n n!} \leq 1, \quad (39)$$

then, for this h , equation (1) is non-oscillatory on the set g_M .

Proof. If all $p_{xk}^h \equiv 0$ ($k = 0, 1, \dots, n - 1; x \in g_{M-n}$), then by Theorem 1 of [1] equation (1) is non-oscillatory on the entire axis. This is also easy to see directly. In what follows we shall assume that at least one of the coefficients of equation (1) is different from zero and, consequently, $L_k^h \neq 0$ for at least one k ($0 \leq k \leq n - 1$).

Suppose that, for the given h , there exists a nontrivial solution u_x^h of equation (1) having n changes of sign on the set g_M . Let x_1 and x_2 ($0 \leq x_1 < x_2 \leq M - 1$) be the first and the last of the points at which u_x^h has changes of sign.

Set

$$A_h = \max_{x=x_1, \dots, x_2-n+1} \left| \frac{\Delta^n u_x^h}{h^n} \right|.$$

Since u_x^h has n changes of sign, it cannot be a polynomial of degree lower than n . Therefore $A_h > 0$. Since g_{M-n} , for fixed h , contains a finite number of points, the maximum

$$\max_{x=x_1, \dots, x_2-n+1} \left| \frac{\Delta^n u_x^h}{h^n} \right|$$

is attained at one of the points x^* of the set $x = x_1, \dots, x_2 - n + 1$, i.e.

$$A_h = \left| \frac{\Delta^n u_{x^*}^h}{h^n} \right|.$$

If

$$\frac{\Delta^n u_x^h}{h^n} \equiv \pm A,$$

then as x^* we choose any point of the set

$x = x_1, \dots, x_2 - n + 1$, distinct from x_1 and $\frac{x_2 + x_1 + 1}{2}$. If $x_2 - x_1 + 1 \geq 2n$, then, by Theorems 1 and 2,

$$\begin{aligned} |u_{x^*}^h| &< \frac{A_h(n-1)^{n-1}(x_2-x_1+1)^n h^n}{n^n n!} \leq \frac{A_h(n-1)^{n-1} M^n h^n}{n^n n!}, \\ \left| \frac{\Delta^k u_{x^*}^h}{h^k} \right| &< \frac{A_h k (x_2 - x_1 - k + 1)^{n-k} h^{n-k}}{n(n-k)!} \leq \frac{A_h k (M-k)^{n-k} h^{n-k}}{n(n-k)!} \\ &(k = 1, \dots, n-1). \end{aligned}$$

It is easy to see that when $x_2 - x_1 + 1 < 2n$ the inequalities

$$\begin{aligned} |u_{x^*}^h| &< \frac{A_h(n-1)^{n-1} M^n h^n}{n^n n!}, \quad \left| \frac{\Delta^k u_{x^*}^h}{h^k} \right| < \frac{A_h k (M-k)^{n-k} h^{n-k}}{n(n-k)!} \\ &(k = 1, \dots, n-1) \end{aligned}$$

are also valid.

Since $L_k^h \neq 0$ for at least one k ($0 \leq k \leq n-1$), from equation (1), in view of the inequalities given above, we obtain

$$\begin{aligned} A_h &= \left| \frac{\Delta^n u_{x^*}^h}{h^n} \right| \leq \sum_{k=0}^{n-1} |p_{x^* k}^h| \left| \frac{\Delta^k u_{x^*}^h}{h^k} \right| < \\ &< \sum_{k=1}^{n-1} L_k^h \frac{A_h k (M-k)^{n-k} h^{n-k}}{n(n-k)!} + L_0^h \frac{A_h (n-1)^{n-1} M^n h^n}{n^n n!}. \end{aligned}$$

Hence

$$\sum_{k=1}^{n-1} \frac{L_k^h k (M-k)^{n-k} h^{n-k}}{n(n-k)!} + \frac{L_0^h (n-1)^{n-1} M^n h^n}{n^n n!} > 1,$$

which contradicts condition (39).

Thus, for the given h , a nontrivial solution of equation (1) cannot have n changes of sign on the set g_M .

The theorem is proved.

Corollary. If there exist constants L_k , independent of h , ($k = 0, 1, \dots, n - 1$), such that for every $h = \frac{b-a}{M} < H$ (H is a known number) the inequalities

$$|p_{xk}^h| \leq L_k \quad (k = 0, 1, \dots, n - 1; x \in g_{M-n}),$$

$$\sum_{k=1}^{n-1} \frac{L_k k (b-a)^{n-k}}{n(n-k)!} + \frac{L_0 (n-1)^{n-1} (b-a)^n}{n^n n!} \leq 1, \quad (40)$$

hold, then $[a, b]$ is a non-oscillation interval of equation (1).

Proof. Since $Mh = b - a$, $(M - k)h < b - a$ ($k = 1, \dots, n - 1$), it follows from inequality (40) that inequality (39) holds for $L_k^h = L_k$ ($k = 0, 1, \dots, n - 1$) and any $h < H$, whence the present assertion follows.

Remark. It is interesting to note the following difference between the non-oscillatory properties of differential and difference equations. Namely, for a linear differential equation, under the usual conditions guaranteeing existence and uniqueness of the solution of the initial-value problem, there always exists an interval of non-oscillation (the Vallee-Poussin theorem), whereas a difference equation, under the condition of existence and uniqueness of the solution of the initial-value problem—consisting in the fact that the coefficients of this equation have finite values for every $x \in g_{M-n}$ and every $h = \frac{b-a}{M}$ —may fail to have an interval of non-oscillation. Indeed, the equation

$$\frac{\Delta^2 u_x^h}{h^2} + \frac{p}{h} u_x^h = 0, \quad (41)$$

where $p = \text{const} > 0$, has the solution

$$u_x^h = (1 + ph)^{\frac{t-a}{2h}} \sin \left\{ \frac{\text{arctg} \sqrt{ph}}{h} (t_x - a) \right\},$$

which, for sufficiently small h , changes sign twice in any interval. Thus, equation (41) has no interval of non-oscillation.

From the corollary to Theorem 3 it is clear that a sufficient condition for the existence of an interval of non-oscillation of equation (1) consists in the uniform boundedness, with respect to h , of its coefficients. However, this condition is not necessary. As shown in [1], for the equation

$$\frac{\Delta^2 u_x^h}{h^2} - K \frac{\Delta u_x^h}{h} + \left(q - \frac{2K}{h} \right) u_x^h = 0$$

with $K > 0$, the interval of non-oscillation is the whole axis, although

$$\lim_{h \rightarrow 0} \left(q - \frac{2K}{h} \right) = -\infty.$$

Let us now consider the n -point difference boundary-value problem

$$N_h[u_x^h] \equiv \frac{\Delta^n u_x^h}{h^n} - f \left(t_x, u_x^h, \frac{\Delta u_x^h}{h}, \dots, \frac{\Delta^{n-1} u_x^h}{h^{n-1}} \right) = 0, \quad (42)$$

$$u_{x_i}^h = A_i \quad (x_i \in g_M; i = 1, \dots, n; 0 = x_1 < \dots < x_n = M). \quad (43)$$

In the form (43), in particular, one may write the boundary conditions

$$\frac{\Delta^{k_j} u_{x_j}^h}{h^{k_j}} = B_{j k_j}, \quad (x_j \in g_M; k_j = 0, 1, \dots, r_j - 1;$$

$$j = 1, \dots, m; \sum_{j=1}^m r_j = n; x_1 < \dots < x_m).$$

Let the function $f(t, y_0, \dots, y_{n-1})$ be defined in the domain $D: a \leq t \leq b, |y_k| \leq C_k$ ($k = 0, 1, \dots, n-1$), and let it satisfy there, with respect to y_0, \dots, y_{n-1} , the Lipschitz condition with constants L_0, \dots, L_{n-1} , respectively.

For problem (42), (43), when h is small, the results of [1, 2] may be inapplicable, since in those works the non-oscillation of certain

auxiliary linear equations, whereas in the case under consideration these equations may have no interval of non-oscillation. However, using the preceding results, for the boundary-value problem (42), (43) one can prove the following theorems.

Theorem 4. Suppose $C_k = \infty$ ($k = 0, 1, \dots, n-1$). If, for the given h ,

$$\sum_{k=1}^{n-1} \frac{L_k k (M-k)^{n-k} h^{n-k}}{n(n-k)!} + \frac{L_0 (n-1)^{n-1} M n h^n}{n^n n!} < 1, \quad (44)$$

then, for this h , the boundary-value problem (42), (43) has a unique solution.

The proof of this theorem is carried out in complete analogy with the proof of Theorem 2 of [3].

In the following theorem no assumptions are made concerning C_k ($k = 0, 1, \dots, n-1$).

Theorem 5. Suppose that, for the given h ,

$$\sum_{k=1}^{n-1} \frac{L_k k (M-k)^{n-k} h^{n-k}}{n(n-k)!} + \frac{L_0 (n-1)^{n-1} M n h^n}{n^n n!} \leq 1. \quad (45)$$

If there exists a solution u_x^h of the problem (42), (43) and a function z_x^h satisfying the boundary conditions (43) and the inequalities

$$\left| \frac{\Delta^k z_x^h}{h^k} \right| \leq C_k \quad (k = 0, 1, \dots, n-1; x \in g_{M-k}),$$

$$N_h[z_x^h] \geq 0 (\leq 0), \quad x \in g_{M-n},$$

then

$$\text{sign}_{x_i < x < x_{i+1}} [z_x^h - u_x^h] = (-1)^{n-i} ((-1)^{n-i+1}) \quad (i = 1, \dots, n-1; x \in g_M)$$

everywhere where $z_x^h \neq u_x^h$.

Proof. Define, on the set g_{M-n} , the functions p_{xk}^h ($k = 0, 1, \dots, n-1$) as follows:

$$p_{xk}^h = \begin{cases} \left[f \left(t_x, u_x^h, \dots, \frac{\Delta^{k-1} u_x^h}{h^{k-1}}, \frac{\Delta^k z_x^h}{h^k}, \dots, \frac{\Delta^{n-1} z_x^h}{h^{n-1}} \right) \right. \\ \left. - f \left(t_x, u_x^h, \dots, \frac{\Delta^k u_x^h}{h^k}, \frac{\Delta^{k+1} z_x^h}{h^{k+1}}, \dots, \frac{\Delta^{n-1} z_x^h}{h^{n-1}} \right) \right] : \left(\frac{\Delta^k z_x^h}{h^k} - \frac{\Delta^k u_x^h}{h^k} \right), \\ \text{for those } x \text{ for which } \frac{\Delta^k z_x^h}{h^k} \neq \frac{\Delta^k u_x^h}{h^k}, \\ 0 \text{ for those } x \text{ for which } \frac{\Delta^k z_x^h}{h^k} = \frac{\Delta^k u_x^h}{h^k}. \end{cases}$$

$$(k = 0, 1, \dots, n-1).$$

It is easy to see that

$$|p_{xk}^h| \leq L_k \quad (k = 0, 1, \dots, n-1; x \in g_{M-n}).$$

Therefore, by virtue of (45) and Theorem 3, the equation

$$L_h[v_x^h] \equiv \frac{\Delta^n v_x^h}{h^n} - \sum_{k=0}^{n-1} p_{xk}^h \frac{\Delta^k v_x^h}{h^k} = 0 \quad (46)$$

is non-oscillatory on the set g_M .

Denote $N_h[z^h] = \varphi_x^h$. Subtracting (42) term by term from this equality and denoting $z_x^h - u_x^h = \eta_x^h$, we obtain

$$L_h[\eta_x^h] = \varphi_x^h \geq 0 (\leq 0), \quad \eta_{x_i}^h = 0 \quad (i = 1, \dots, n).$$

Hence, by virtue of the non-oscillation of equation (46) and the results of [1], it follows that

$$\operatorname{sign}_{x_i < x < x_{i+1}} \eta_x^h = (-1)^{n-i} ((-1)^{n-i+1}) \quad (i = 1, \dots, n-1; x \in g_M)$$

everywhere where $\eta_x^h \neq 0$.

The theorem is proved.

Theorem 6. Let $C_0 < \infty$, $C_k = \infty$ ($k = 1, \dots, n-1$), and suppose that, for the given h , inequality (44) is satisfied.

If, moreover, for this h there exist functions z_{xj}^h ($j = 1, 2$), defined on the set g_M and satisfying the boundary conditions (43) and the inequalities

$$|z_{xj}^h| \leq C_0 \quad (j = 1, 2; x \in g_M), \quad (47)$$

$$N_h[z_{x1}^h] \geq 0, \quad N_h[z_{x2}^h] \leq 0 \quad (x \in g_{M-n}), \quad (48)$$

then the problem (42), (43) has a unique solution u_x^h , and

$$\operatorname{sign}_{x_i < x < x_{i+1}} [z_{xj}^h - u_x^h] = (-1)^{n-i+j-1} \quad (i = 1, \dots, n-1; j = 1, 2; x \in g_M)$$

everywhere where $z_{xj}^h \neq u_x^h$ ($j = 1, 2$).

Proof. In the domain $D^* : a \leq t \leq b$, $|y_k| < \infty$ ($k = 0, 1, \dots, n-1$), define the function $f^*(t, y_0, \dots, y_{n-1})$ by the equalities

$$f^*(t, y_0, \dots, y_{n-1}) = \begin{cases} f(t, y_0, \dots, y_{n-1}) & \text{for } |y_0| \leq C_0, \\ f(t, C_0, y_1, \dots, y_{n-1}) + \lambda(t)(y_0 - C_0) & \text{for } y_0 > C_0, \\ f(t, -C_0, y_1, \dots, y_{n-1}) + \lambda(t)(y_0 + C_0) & \text{for } y_0 < -C_0, \end{cases}$$

where $\lambda(t)$ is a function satisfying the inequality $|\lambda(t)| \leq L_0$ ($t \in [a, b]$). Obviously, $f^*(t, y_0, \dots, y_{n-1})$ in the domain D^* satisfies, with respect to y_0, \dots, y_{n-1} , a Lipschitz condition with constants L_0, \dots, L_{n-1} , respectively.

The boundary-value problem

$$N_h^*[v_x^h] \equiv \frac{\Delta^n v_x^h}{h^n} - f^* \left(t_x, v_x^h, \frac{\Delta v_x^h}{h}, \dots, \frac{\Delta^{n-1} v_x^h}{h^{n-1}} \right) = 0,$$

$$v_{x_i}^h = A_i \quad (i = 1, \dots, n)$$

has, by virtue of Theorem 4, a unique solution v_x^h .

Since, for $|y_0| \leq C_0$, the function $f^*(t, y_0, \dots, y_{n-1})$ coincides with $f(t, y_0, \dots, y_{n-1})$, it follows from (47) and (48) that

$$N_h^*[z_{x1}^h] \geq 0, \quad N_h^*[z_{x2}^h] \leq 0 \quad (x \in g_{M-n}).$$

Hence, by Theorem 5,

$$\operatorname{sign}_{x_i < x < x_{i+1}} [z_{xj}^h - v_x^h] = (-1)^{n-i+j-1} \quad (i = 1, \dots, n-1; j = 1, 2; x \in g_M). \quad (49)$$

everywhere where $z_{xj}^h \neq v_x^h$ ($j = 1, 2$). Hence, by virtue of (47), $|v_x^h| \leq C_0$ ($x \in g_M$), and therefore

$$f^* \left(t_x, v_x^h, \frac{\Delta v_x^h}{h}, \dots, \frac{\Delta^{n-1} v_x^h}{h^{n-1}} \right) = f \left(t_x, v_x^h, \frac{\Delta v_x^h}{h}, \dots, \frac{\Delta^{n-1} v_x^h}{h^{n-1}} \right) \quad (x \in g_{M-n}),$$

i.e., v_x^h is a solution of problem (42), (43), and this solution is unique and satisfies relations (49).

The theorem is proved.

Following [8], let us call equation (42) with boundary conditions (43) stable with respect to the right-hand side if, for $h < h_0$ ($h_0 > 0$ is some number), it has a solution u_x^h , and for every $\varepsilon > 0$ there exists a $\delta > 0$, independent of h , such that for any z_x^h satisfying the boundary conditions (43), for arbitrary $h < h_0$, the inequality

$$\max_{x \in g_M} |z_x^h - u_x^h| < \varepsilon, \quad (50)$$

holds as soon as

$$\max_{x \in g_{M-n}} |N_h[z_x^h]| < \delta. \quad (51)$$

Theorem 7. If $C_k = \infty$ ($k = 0, 1, \dots, n-1$) and

$$\sum_{k=1}^{n-1} \frac{L_k k (b-a)^{n-k}}{n(n-k)!} + \frac{L_0 (n-1)^{n-1} (b-a)^n}{n^n n!} < 1, \quad (52)$$

then equation (42) with boundary conditions (43) is stable with respect to the right-hand side.

Proof. Since $Mh = b - a$, $(M - k)h < b - a$ ($k = 1, \dots, n-1$), it follows from inequality (52), for any h , that inequality (44) holds. Therefore, by Theorem 4, the boundary-value problem (42), (43) has, for any h , a unique solution u_x^h .

Suppose that, for some $\varepsilon = \varepsilon_1 > 0$ and any $\delta > 0$, there exists a $z_{x\delta}^h$ satisfying conditions (43), and an $h(\delta)$, such that for $h = h(\delta)$

$$\max_{x \in g_M} |z_{x\delta}^{h(\delta)} - u_x^{h(\delta)}| \geq \varepsilon_1,$$

despite the fulfillment of inequality (51). Denote

$$\max_{x \in g_{M-n}} \left| \frac{\Delta^n (z_{x\delta}^{h(\delta)} - u_x^{h(\delta)})}{[h(\delta)]^n} \right| = A_\delta,$$

$$N_h[z_{x\delta}^h] = \varphi_{x\delta}^h. \quad (53)$$

Since $z_{x\delta}^h - u_x^h$ has n zeros on g_M , including at the points $x = 0$ and $x = M$, it follows from Theorem 1 that

$$\begin{aligned} \varepsilon_1 &\leq \max_{x \in g_M} |z_{x\delta}^{h(\delta)} - u_x^{h(\delta)}| \leq \frac{A_\delta (n-1)^{n-1} M^n [h(\delta)]^n}{n^n n!} = \\ &= \frac{A_\delta (n-1)^{n-1} (b-a)^n}{n^n n!}, \end{aligned}$$

whence

$$A_\delta \geq \frac{\varepsilon_1 n^n n!}{(n-1)^{n-1} (b-a)^n}. \quad (54)$$

Subtracting (42) term by term from (53), by virtue of the Lipschitz condition and Theorems 1 and 2, for $h = h(\delta)$ we obtain

$$\begin{aligned}
 A_\delta &= \max_{x \in g_{M-n}} \left| \frac{\Delta^n(z_{x0}^{h(\delta)} - u_x^{h(\delta)})}{[h(\delta)]^n} \right| \leq \sum_{k=0}^{n-1} L_k \max_{x \in g_{M-n}} \left| \frac{\Delta^k(z_{x0}^{h(\delta)} - u_x^{h(\delta)})}{[h(\delta)]^k} \right| \\
 &+ \max_{x \in g_{M-n}} |\varphi_{x0}^{h(\delta)}| < \sum_{k=1}^{n-1} \frac{L_k A_\delta k (M-k)^{n-k} [h(\delta)]^{n-k}}{n(n-k)!} \\
 &+ \frac{L_0 A_\delta (n-1)^{n-1} M^n [h(\delta)]^n}{n^n n!} + \delta \leq \sum_{k=1}^{n-1} \frac{L_k A_\delta k (b-a)^{n-k}}{n(n-k)!} \\
 &+ \frac{L_0 A_\delta (n-1)^{n-1} (b-a)^n}{n^n n!} + \delta,
 \end{aligned} \tag{55}$$

since, according to the assumption,

$$\max_{x \in g_{M-n}} |\varphi_{x0}^{h(\delta)}| < \delta.$$

From (54) and (55) it follows that

$$1 < \sum_{k=1}^{n-1} \frac{L_k k (b-a)^{n-k}}{n(n-k)!} + \frac{L_0 (n-1)^{n-1} (b-a)^n}{n^n n!} + \frac{\delta (n-1)^{n-1} (b-a)^n}{\varepsilon_1 n^n n!}$$

for arbitrarily small δ and fixed ε_1 . Hence,

$$\sum_{k=1}^{n-1} \frac{L_k k (b-a)^{n-k}}{n(n-k)!} + \frac{L_0 (n-1)^{n-1} (b-a)^n}{n^n n!} \geq 1,$$

which contradicts condition (52). Consequently, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each h , inequality (51) implies inequality (50), provided only that z_x^h satisfies conditions (43).

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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