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Abstract

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CLASSIFICATION OF IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF SECOND-ORDER MATRICES WITH ELEMENTS FROM A LOCALLY COMPACT FIELD

(Presented by Academician I. G. Petrovskii, 26 VIII 1965)

Notation: K is a disconnected locally compact nondiscrete field; G is the group of nonsingular second-order matrices with elements from K; $\widetilde{G}$ is the subgroup of matrices with determinant 1; G_0 is the subgroup of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix};$$

Π is the set of unitary multiplicative characters of the field K; $\nu(x)$ is the norm of the element $x \in K$.

In the paper $(^1)$ a construction was given of several series of irreducible unitary representations of the group G, and it was shown that these representations suffice for decomposing the regular representation of G into irreducible components. The question of the existence of other irreducible representations remained open. The purpose of the present note is to show that the group G has no irreducible unitary representations except those constructed in $(^1)$. It will be more convenient for us to prove the analogous assertion for the group $\widetilde{G}$. Let us list the irreducible representations of $\widetilde{G}$ that can be constructed on the basis of the results of $(^1)$. By g we denote the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \widetilde{G}.$$

1. The continuous series consists of representations T_{π_1,π_2}, $\pi_1, \pi_2 \in \Pi$, acting in the space $L^2(K, dx)$ according to the formula

$$T_{\pi_1,\pi_2}(g)f(x) = \pi_1(\beta x + \delta)\, \pi_2\left(\frac{\alpha\delta - \beta\gamma}{\beta x + \delta}\right) \frac{\nu^{1/2}(\alpha\delta - \beta\gamma)}{\nu(\beta x + \delta)} f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

2. The supplementary series consists of representations $V_{\pi_0,\rho}$, $\pi_0 \in \Pi$, $0 < \rho < 1$, acting in the space of functions on K with scalar product

$$(f_1, f_2) = \iint_{K\,K} f_1(x)\,\overline{f_2(y)}\,\nu^{-2\rho}(x-y)\,dx\,dy.$$

The operators of the representation have the form

$$V_{\pi_0,\rho}(g)f(x) = \pi_0(\alpha\delta-\beta\gamma)\,\frac{\nu^{1-\rho}(\alpha\delta-\beta\gamma)}{\nu^{2-2\rho}(\beta x+\delta)}\,f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right).$$

3. The special series consists of representations S_{π_0}, $\pi_0 \in \Pi$, acting in the space of functions on K for which $\int f(x)\,dx = 0$; the scalar product is given by the formula

$$(f_1, f_2) = \iint_{K\,K} f_1(x)\,\overline{f_2(y)}\,\ln\nu(x-y)\,dx\,dy.$$

The operators of the representation have the form

$$S_{\pi_0}(g)f(x) = \pi_0(\alpha\delta-\beta\gamma)\,\frac{\nu(\alpha\delta-\beta\gamma)}{\nu^{2}(\beta x+\delta)}\,f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right).$$

4. The discrete series consists of representations $U_{\tilde{\pi},\pi_0}$, where $\pi_0 \in \Pi$, $\tilde{\pi}$ is a unitary multiplicative character of the field $\widetilde{K} = K(\sqrt{\tau})$.

The representation acts in the space $L^2(K, dx)$ by the formula

$$U_{\tilde{\pi},\pi_0}(g)f(x) = \int_K \mathcal{K}(g \mid x, y) f(y)\,dy;$$

the kernel $\mathcal{K}(g \mid x, y)$ has the form

$$\mathcal{K}(g \mid x, y) = \begin{cases} c\pi_0(\alpha\delta-\beta\gamma)\dfrac{\nu(\beta)}{\nu^{1/2}(\alpha\delta-\beta\gamma)} \displaystyle\int_{K(\sqrt{\tau})} \dfrac{\operatorname{sign}_\tau A}{\nu^2(A)}\,\tilde{\pi}(t)\,dt, & \text{for } \beta \neq 0,\\ \operatorname{sign}_\tau(\delta)\tilde{\pi}(\delta)\nu^{1/2}(\alpha\delta)\delta(\alpha x-\delta y+\gamma), & \text{for } \beta = 0,\end{cases}$$

where

$$A = \beta x + \delta - t - \bar{t} + \frac{\alpha-\beta y}{\alpha\delta-\beta\gamma}\,t\bar{t}.$$

5. The degenerate series consists of one-dimensional representations W_{π_0}, $\pi_0 \in \Pi$,

$$W_{\pi_0}(g) = \pi_0(\alpha\delta - \beta\gamma).$$

Theorem. The representations listed above exhaust the set of all irreducible unitary representations of the group $\widetilde{G}$.

Proof scheme. Let us first consider a finite-dimensional representation T of the group $\widetilde{G}$.

Denote, as in (2), by C_n the subgroup of matrices congruent to E modulo $\mathfrak{p}^n\mathcal{O}$. The operators

$$P_n = \int_{C_n} T(g)\,dg$$

are, evidently, self-adjoint projection operators in the space H of the representation T. Moreover, since the subgroups C_n form a complete system of neighborhoods of the identity in $\widetilde{G}$, the sequence $\{P_n\}$ converges strongly to the identity operator.

In a finite-dimensional space this is possible only if, starting from some n, the equality $P_n = E$ holds. But this means that all vectors of H are invariant with respect to C_n. The kernel of the representation T is a normal divisor in $\widetilde{G}$ and contains the subgroup C_n. It follows that the kernel of T contains the whole subgroup G. Thus T is in fact a representation of the factor group $\widetilde{G}/G$.

This last group is commutative and isomorphic to the multiplicative group of the field K. Thus, all finite-dimensional unitary irreducible representations of the group $\widetilde{G}$ are exhausted by the representations W_{π_0} indicated above, which constitute the degenerate series.

Now let the representation T be infinite-dimensional and irreducible. In [3] we showed that the restriction of T to G_0 is also irreducible and coincides with some fixed representation G_0.

Since the group $\widetilde{G}$ is generated by the subgroup G_0 and the matrix

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

in order to define the representation it suffices to specify the operator $T(s)$. This operator, as shown in (2), has the form

$$T(s)\varphi(\pi) = s(\pi)\varphi(\pi_0\pi^{-1}), \tag{1}$$

where $s(\pi)$ is a function on Π, taking complex values, of modulus equal to 1, and satisfying the functional equation

$$s(\pi_1)\Gamma(\pi_1, \pi_2\pi_0^{-1})s(\pi_2) = \pi_1\pi_2(-1)\int_{\Pi} \Gamma(\pi\pi_1^{-1})s(\pi)\Gamma(\pi\pi_2^{-1})\, d\pi. \qquad (2)$$

The character π_0 in formulas (1) and (2) is determined by the equality $T(d_\lambda) = \pi_0(\lambda)E$, where

$$d_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

We give explicit expressions for the functions $s(\pi)$ corresponding to the series of representations of $\widetilde{G}^*$ listed above.

* We note that from these formulas the equivalence of the representations T_{π_1,π_2} and T_{π_2,π_1}, and the equivalence of the representations $V_{\pi_0,\rho}$ and $V_{\pi_0,1-\rho}$, are immediately apparent.

1. $T = T_{\pi_1,\pi_2}$ —a representation of the noncontinuous series

$$s(\pi) = \pi_1\pi_2(-1)\Gamma(\pi^{-1}\pi_1^{-1}\nu^{1/2})\Gamma(\pi^{-1}\pi_2^{-1}\nu^{1/2}).$$

2. $T = V_{\pi_0,\rho}$ —a representation of the complementary series

$$s(\pi) = \Gamma(\pi^{-1}\pi_0^{-1}\nu^{\rho})\Gamma(\pi^{-1}\pi_0^{-1}\nu^{1-\rho}).$$

3. $T = S_{\pi_0}$ —a representation of the special series

$$s(\pi) = \Gamma(\pi^{-1}\pi_0^{-1})\Gamma(\pi^{-1}\pi_0^{-1}\nu).$$

4. $T = U_{\tilde{\pi},\pi_0}$ —a representation of the discrete series

$$s(\pi) = c\,\Gamma_\tau(\pi\tau),$$

where Γ_τ is the gamma-function of the field $K(\sqrt{\tau})$; π_τ is a nonunitary multiplicative character of this field, given by the formula

$$\pi_\tau(t) = \tilde{\pi}(t) \cdot \pi\pi_0(t^{-1}\bar{t}^{-1}) \cdot \nu^{1/2}(t\bar{t}).$$

For the proof of the theorem it is enough to verify that the functional equation (2) has no other solutions besides those listed above. This is done as follows.

The multiplicative group of the field K is the direct product of an infinite cyclic group with generator p and the compact group $O^* = \{x : \nu(x) = 1\}$. Therefore the set Π is the product of a circle and a discrete set $\Theta = \widehat{O^*}$. Thus one may specify an element $\pi \in \Pi$ by two coordinates (λ, θ), where λ is a complex number of modulus 1, and θ is a character of the group O^*.

Expand the function $s(\pi)$ in a Laurent series in λ:

$$s(\pi) = s(\lambda, \theta) = \sum_{-\infty}^{\infty} s_k(\theta)\lambda^k.$$

The functional equation (2) can be rewritten in the form*

$$(1-\nu(p))\sum_m s_k(\theta_1)\Gamma_{-m}(\theta_1\theta_2\theta_0^{-1})\lambda_0^m s_l(\theta_2) =$$

$$= \theta_1\theta_2(-1)\sum_m \Gamma_{-k}(\theta\theta_1^{-1})s_{k+l}(\theta)\Gamma_{-l}(\theta\theta_2^{-1}), \tag{3}$$

where Γ_k are the coefficients of the Laurent series for the Γ-function of the field K. It turns out that if among the coefficients $s_k(\theta)$, for $k \leq 0$, there is at least one different from zero, then equation (3) is solved explicitly. All such solutions are related to representations of the principal, complementary, and special series. The remaining solutions cannot be written explicitly, but it can be shown that, for the representations corresponding to these solutions, the matrix elements have summable square on the subgroup G. Hence it follows that the corresponding representations enter as discrete summands in the decomposition of $L^2(G)$ into irreducible components. Since in paper (1) a Plancherel formula was obtained giving a decomposition of $L^2(G)$ over representations of the principal and discrete series, we see that the remaining solutions of our functional equation are related to representations of the discrete series.

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CITED LITERATURE

[1] I. M. Gel' fand, M. I. Graev, *UMN*, 18, no. 4, 29 (1963). [2] A. A. Kirillov, *DAN*, 150, no. 4 (1963).

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* We note that in paper (2) the factors $1-\nu(p)$ and λ_0^m are omitted in this equation.

Note: Figure translations are in progress. See original paper for figures.