A METHOD FOR INVESTIGATING THE STABILITY OF SYSTEMS WITH AFTEREFFECT

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Abstract

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MATHEMATICS

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A METHOD FOR INVESTIGATING THE STABILITY OF SYSTEMS WITH AFTEREFFECT

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Let the system of differential equations of the perturbed motion have the form ([1])

$$dx_i/dt = X_i(t; x_1(t-\tau), \dots, x_n(t-\tau)) \quad (i = 1, \dots, n), \tag{1}$$

where $X_i(t; x_1(t-\tau), \dots, x_n(t-\tau))$ are continuous functionals, defined on piecewise-continuous functions $x_i(\sigma)$ $(i = 1, \dots, n)$ on the interval $t - h \le \sigma \le t$ and satisfying the condition

$$X_i(t; 0, \dots, 0) \equiv 0 \quad (i = 1, \dots, n).$$

Simultaneously with (1), consider the system with advance corresponding to this system. Let $t^* > t_0$ be a parameter. Introduce a new independent variable ξ, $t = t^* - \xi$. Noting that $dx/dt = -dx/d\xi$ and making the change of variables, we obtain

$$dx_i(t^* - \xi)/d\xi = -X_i(t^* - \xi)x_1(t^* - \xi - \tau), \dots, x_n(t^* - \xi - \tau))$$

$$(i = 1, \dots, n).$$

Denoting

$$x_i(t^* - \xi) = y_i(\xi), \qquad X_i(t^* - \xi) = -Y_i(\xi), \tag{2}$$

we shall have

$$dy_i/d\xi = Y_i(\xi; y_1(\xi + \tau), \dots, y_n(\xi + \tau)) \quad (i = 1, \dots, n). \tag{3}$$

Obviously, every solution of system (1), corresponding to the system of initial functions $\varphi_1(s), \dots, \varphi_n(s)$ $(t_0 - h \leq s \leq t_0)$, transformed to the independent variable ξ, will be a solution of system (3) in the interval $0 < \xi < t^* - t_0$. Let $V(t; x_1, \dots, x_n)$ be a positive definite function of the variables $x_1, \dots, x_n$, having continuous partial derivatives with respect to all arguments. Its derivative, by virtue of system (1),

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i = U(t; x_1(t-\tau), \dots, x_n(t-\tau)) \tag{4}$$

is a continuous functional, defined on the solutions of system (1) arriving at the point $x_1(t), \dots, x_n(t)$. Thus, the derivative of the function V, by virtue of system (1), at the point $x_1(t), \dots, x_n(t)$ is not single-valued and has as many values as there are integral curves arriving at this point. Making in the functional U the substitution of the independent variable, we obtain the functional

$$W(\xi; y_1(\xi+\tau), \dots, y_n(\xi+\tau)) =$$

$$= U(t^* - \xi; x_1(t^* - \xi - \tau), \dots, x_n(t^* - \xi - \tau)), \tag{5}$$

defined on the solutions of the system with advance (3).

The extension of the Lyapunov function method ([2]) to systems of the form (1) is given by the following theorems:

Theorem 1. If, for the differential equations of the perturbed motion (1), there exists a positive definite function $V(t; x_1, \dots, x_n)$, whose derivative dV/dt by virtue of this system is such that the functional $W(0; y_1(\tau), \dots, y_n(\tau))$ is negative or identically equal to zero along every continuous solution of system (3) satisfying the conditions

$$y_i(0) = x_i \qquad (i = 1, \dots, n), \tag{6}$$

$$V(t-\sigma; y_1(\sigma), \dots, y_n(\sigma)) \leq V(t; x_1, \dots, x_n) \quad \text{for } \sigma \leq t, \quad 0 \leq \sigma \leq t - t_0, \tag{7}$$

then the unperturbed motion is stable.*

Proof. Let ε be an arbitrarily given positive number less than $\bar{c} = \inf V(t; x_1, \dots, x_n)$ for $t \geq t_0$ and $x_1, \dots, x_n$ belonging to the boundary of the domain $|x_s| \leq H$, in which the conditions of the existence and uniqueness theorem for system (1) are satisfied ([3–5]). Let

$$c = \inf V(t; x_1, \dots, x_n)$$

on the sphere

$$\sum_{s=1}^{n} x_s^2 = \varepsilon$$

and $t \geq t_0$, and let

$$\delta(\varepsilon) = \inf \sum_{s=1}^{n} x_s^2$$

on the set $t \geq t_0$, $V(t; x_1, \dots, x_n) = c$. Consider the solution $x_1(t), \dots, x_n(t)$ corresponding to an arbitrary system of initial functions $\varphi_1(s), \dots, \varphi_n(s)$ satisfying the condition

$$\sum_{i=1}^{n} \varphi_i^2(s) \leq \delta_1(\varepsilon),$$

where $\delta_1 < \delta(\varepsilon)$ is so small that the corresponding solution does not leave the domain

$$\sum_{i=1}^{n} x_i^2(t) \leq \delta(\varepsilon)$$

in the time interval $t_0 \leq t \leq t_0 + h$. Suppose that the assertion of the theorem is false and that there exists at least one system of initial functions such that

$$\sum_{i=1}^{n} \varphi_i^2 < \delta_1(\varepsilon)$$

and such that the corresponding solution leaves the sphere

$$\sum_{i=1}^{n} x_i^2 = \varepsilon,$$

i.e., there exists a time $t_1 > t_0 + h$ such that

$$\sum_{i=1}^{n} x_i^2(t) \begin{cases} < \varepsilon, & \text{for } t < t_1, \\ > \varepsilon, & \text{for } t_1 - \lambda > t > t_1, \end{cases} \tag{8}$$

where λ is a positive number different from zero, or ∞. Then, since

$$c = \inf V$$

on

$$\sum_{i=1}^{n} x_i^2 = \varepsilon,$$

there exists a time t_2, $t_0 + h \leq t_2 \leq t_1$, such that along this solution

$$V(t; x_1(t), \dots, x_n(t)) \begin{cases} < c, & \text{for } t < t_2, \\ > c, & \text{for } t_2 + \mu > t > t_2, \end{cases} \tag{9}$$

where μ is a positive number different from zero, or ∞. Condition (9) means that

$$\left.\frac{dV}{dt}\right|_{t=t_2} > 0,$$

but the integral curve under consideration, under reversal of the independent variable, is an integral curve of system (3), satisfying conditions (6) and (7), and, consequently, along this curve the inequality

$$\left.\frac{dV}{dt}\right|_{t=t_2} > 0$$

is impossible. The theorem is proved.

* The definitions of the concepts used in the paper are given in ([1]).

The following theorem on asymptotic stability is also valid:

Theorem 2. *If for (1) the conditions of Theorem 1 are satisfied and the function V admits an infinitely small upper limit, while the functional $W(0; y_1(\tau), \dots, y_n(\tau))$ is negative definite under conditions (6) and (7), then the unperturbed motion is asymptotically stable.*

The proof of this theorem is analogous to the proof of the theorem on asymptotic stability in (1).

A simpler problem is the problem of integrating, or estimating, the set of solutions of system (3) satisfying conditions (6) and (7) on an interval of fixed length $T \geqslant \sigma \geqslant 0$.

Theorems 1 and 2, under the condition $T \geqslant \sigma \geqslant 0$, give sufficient conditions for stability and asymptotic stability, the crudeness of which is determined by the magnitude of T.

To estimate the set of solutions satisfying conditions (6) and (7) and $T \geqslant \sigma \geqslant 0$, it is possible to use the method of steps. Indeed, let $\varphi_1(s), \dots, \varphi_n(s)$ be a system of continuous initial functions given on the interval $t - T - h \leqslant s \leqslant t - T$ $(T > h,\ t > t_0 + T)$ and satisfying the condition $V(s; \varphi_1(s), \dots, \varphi_n(s)) \leqslant V(t; x_1(t), \dots, x_n(t))$. The corresponding solution on the interval $(t - T, t)$ can be found by the method of steps. Let $E_0(t; T)$ be the set of systems of initial functions satisfying the condition $V(s; \varphi(s)) \leqslant V(t; x(t))$, and let $E(t; T)$ be the set of corresponding solutions. Let $E'(t; T) \subset E(t; T)$ be the subset of solutions $x_1^*(\sigma), \dots, x_n^*(\sigma)$ satisfying the conditions

$$V(\sigma; x_1^*(\sigma), \dots, x_n^*(\sigma)) \leqslant V(t; x_1(t), \dots, x_n(t)), \quad t - T \leqslant \sigma \leqslant t,$$

$$x_i^*(t) = x(t).$$

In this case, Theorem 1 implies the validity of the assertion:

Theorem 3. *If for the differential equations of the perturbed motion there exists a positive number $T > h$ and a positive definite function $V(t; x_1, \dots, x_n)$, whose derivative by virtue of this system $dV/dt = U(t; x_1(t - \tau), \dots, x_n(t - \tau))$ is a negative, or identically zero, functional on the set of functions $E'(t; T)$ for every $t > t_0 + T$, then the unperturbed motion is stable.*

Remark. The set $E_0(t; T)$, without loss of generality, may be restricted to functions of class C_k, where k is an arbitrarily large fixed integer, since $\varphi_1(s), \dots, \varphi_n(s)$ for $t - T - h \leqslant s \leqslant t - T$ are in fact solutions of system (1) and, for sufficiently large t, have continuous derivatives up to order k inclusive on the interval $t - T - h \leqslant s \leqslant t - T$.

To solve the stability problem for a system with aftereffect, it is necessary to find conditions under which the functional $dV/dt = U(t; x_1(t-\tau), \dots, x_n(t-\tau))$ is negative on the set of integral curves of system (3) satisfying conditions (6) and (7) of Theorem 1. Obviously, the functional $W(0; y_1(\tau), \dots, y_n(\tau))$ will be negative if the maximum of this functional on the set of solutions of system (3) is a sign-definite function of the variables $x_1, \dots, x_n$.

Indeed, the value of the functional on a given integral curve of system (3) is a function only of the coordinates $x_1, \dots, x_n$, as is the maximum of this functional on the set of solutions of system (3) satisfying conditions (6) and (7).

Thus, the problem of determining the conditions under which the given positive definite function $V(t; x_1, \dots, x_n)$ satisfies the conditions of Theorem 1 reduces to the problem of the maximum of the functional W on the set of integral curves of system (3) satisfying conditions (6) and (7).

Theorem 3 makes it possible to formulate a simpler variational problem, whose solution will make it possible to indicate sufficient conditions for stability. Indeed, in this case the problem of the maximum is posed

of the functional $U(\xi, x_1(\xi-\tau), \dots, x_n(\xi-\tau))$ on the set E', which includes the set of integral curves of system (3) satisfying conditions (6) and (7) of Theorem 1. What is essential in the case of applying Theorem 3 is that it is possible to use the method of steps. Indeed, in cases where the method of steps makes it possible to find an expression for the shift operator over an interval of length T, it makes it possible to find an explicit expression for the functional U in terms of the initial functions $\varphi_1(\xi), \dots, \varphi_n(\xi)$, specified on the interval $t-T-h \leqslant \xi \leqslant t-T$. In this case we obtain the problem of the maximum of the functional $U(t; x_1(t), \dots, x_n(t), \varphi_1(\xi), \dots, \varphi_n(\xi))$ with constraints of the form

$$V(\xi; \varphi_1(\xi), \dots, \varphi_n(\xi)) \leqslant V(t; x_1(t), \dots, x_n(t)),$$

$$|d\varphi_i/d\xi| \leqslant \sup |X_i(t; y_1(t-\tau), \dots, y_n(t-\tau))|$$

on the set defined by the conditions

$$V(t; y_1(\xi), \dots, y_n(\xi)) \leqslant V(t; x_1(t), \dots, x_n(t)), \qquad \xi \leqslant t, \qquad y_i(t) = x_i(t).$$

For the solution of such a problem, the known methods ([6], [7]) may be used.

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Note: Figure translations are in progress. See original paper for figures.