ON A PROPERTY OF SOLUTIONS OF A PARABOLIC EQUATION

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Abstract

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ON A PROPERTY OF SOLUTIONS OF A PARABOLIC EQUATION

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For linear parabolic equations of the second order

$$u_t = \sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} + \sum_{i=1}^{n} b_i u_{x_i} + cu \tag{1}$$

with coefficients depending only on x, as Ito and Yamabe showed ([1]), the following assertion is valid: if u is a solution of such an equation, defined in the cylinder $x \in D,\ 0 \leq t \leq T$, vanishing on the lateral surface of the cylinder and, moreover, vanishing on some part of the upper base of the cylinder which is an n-dimensional domain, then $u \equiv 0$.

The independence of the coefficients from time is essential in the considerations of these authors. Does this fact remain valid if in equation (1) the coefficients are also allowed to depend on t?

In Theorem 1 of the present note a positive answer to this question is given for the case when there is one spatial variable. Closely connected with this question is another: suppose that a solution of the parabolic equation (1) vanishes in some n-dimensional domain lying on a characteristic. Will the solution be identically zero on the whole characteristic? A partial answer to this question, again for the case of one spatial variable, is given by Theorem 2.

Thus, let us consider the equation

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u, \tag{2}$$

concerning whose coefficients we shall assume: a and b are continuously differentiable, c is bounded, and the inequalities $a(t,x) > a_0 > 0,\ c \leq 0$ hold.

Theorem 1. Let D be the domain bounded by the straight lines $t = 0,\ t = T > 0$ and the curves $x = \varphi_1(t),\ x = \varphi_2(t),\ \varphi_2(t) > \varphi_1(t),\ 0 \leq t \leq T$. Let $u(t,x)$ be a solution of equation (2) in the domain D, continuous in the closed domain and satisfying the conditions

$$u\big|_{x=\varphi_1(t)} = u\big|_{x=\varphi_2(t)} = 0,$$

$$u(T,x) = 0 \quad \text{for } a < x < b, \quad \varphi_1(T) \le a < b \le \varphi_2(T).$$

Then $u \equiv 0$ in D.

Theorem 2. Let $u(t,x)$ be a solution of equation (2) in D, continuous in the closed domain and vanishing on the segment $a \le x \le b$, $\varphi_1(T) < a < b < \varphi_2(T)$, of the straight line $t = T$ (the domain D is the same as in the preceding theorem). Then $u \equiv 0$ at least on one of the segments $[\varphi_1(T), a]$ or $[b, \varphi_2(T)]$ of the straight line $t = T$.

The proof of these theorems is based on the following lemmas.

Lemma 1. Let G be a bounded domain situated in the strip between the straight lines $t = t_1$ and $t = t_2$, $t_2 > t_1$, and having limit points on both sides of this strip. Denote by Γ that part of the boundary of the domain

G, which lies strictly inside the strip. Suppose that in G there is defined a solution of equation (2), continuous in the closed domain and vanishing on Γ. Then the inequality

$$\max_x |u(t_1,x)| \ge 2^{-\eta(t_2-t_1)^3/(\mu_2 G)^2} \max_x |u(t_2,x)|,$$

holds, where $\eta > 0$ is a constant depending on the coefficients of the equation and on the length of the projection of the domain G onto the x-axis.

By $\mu_2 A$ we denote the planar Lebesgue measure of a set A lying in the plane; by $\mu_1 B$ we denote the linear Lebesgue measure of a set B lying on a line.

The proof of this lemma can be found in my paper ([2]) (Theorem 2.3.1, p. 71).

Lemma 2. Let $T < 1$ be some positive number. Let Π be the strip in the (t,x)-plane enclosed between the straight lines $t = \sqrt{T}\,x$ and $t = \sqrt{T}\,x + T$ (or $t = -\sqrt{T}\,x$ and $t = -\sqrt{T}\,x + T$). Suppose that the boundaries of this strip are joined by a curve Γ, projecting one-to-one onto the t-axis and located below the x-axis (we assume that the t-axis is directed upward, and the x-axis to the right). Denote by G the part of the strip between the curve Γ and the x-axis. Suppose that in G there is defined a solution $u(t,x)$ of equation (2), continuous in $\overline{G}$ and possessing the following properties: $u|_{t=0} = 0$, $u|_\Gamma = u_0 > 0$, and $u > -u_0$ in G.

Denote by G' the set of points $(t,x) \in G$ at which $u_0/2 < u(t,x) < u_0$. Then $\mu_2 G' > \eta T^{3/2}$, where η is a constant depending on the coefficients of the equation.

This lemma, in turn, is proved with the aid of the following two lemmas:

Lemma 3. Let the domain G be the same as in Lemma 2, and suppose that in it there is defined a solution $u(t,x)$ of equation (2) with the same properties as in Lemma 2. Denote by E_{u^*} the projection onto the t-axis of the set of points (t,x) belonging to the intersection of G with the strip

$$\sqrt{T}\,x + T/3 < t < \sqrt{T}\,x + 2T/3,$$

at which $u(t,x) = u^*$. There exist two constants $\eta_1 > 0$ and $\eta_2 > 0$, depending on the coefficients of the equation, such that from the fact that the set of those values $u^* \in [u_0/2, u_0]$ for which $\mu_1 E_{u^*} < \eta_1 T$ forms a set whose measure is greater than $u_0/4$, it follows that $\mu_2 G' > \eta_2 T^{3/2}$, where G' has the same meaning as in Lemma 2.

Lemma 4. Suppose that in the rectangle $0 < t < \tau,\ 0 < x < \sqrt{\tau}$, $\tau < 1$, there is defined a solution $u(t,x)$ of equation (2) satisfying the condition: $u(t,x) > -u_0$, where $u_0 > 0$. Denote by H_{u^*} the projection onto the t-axis of the set of points (t,x) belonging to the rectangle $0 < t < \tau,\ \frac{1}{4}\sqrt{\tau} < x < \frac{3}{4}\sqrt{\tau}$, at which $u(t,x) = u^*$. Whatever positive number η_3 may be, there exists a positive number η_4, depending on η_3 and on the coefficients of the equation, such that from the fact that the set of those values $u^* \in [u_0/2, u_0]$ for which $\mu_1 H_{u^*} \geq \eta_4 \tau$ forms a set whose measure is not less than $u_0/4$, it follows that for the set G' of points (t,x), where $u_0/2 < u(t,x) < u_0$, the inequality $\mu_2 G' > \eta_4 T^{3/2}$ is valid.

The proof of these lemmas is obtained with the aid of Green' s formula. The proof of Lemma 4 has much in common with the proof of Lemma 2.4.1 of paper ([2]) (p. 73).

Proof of Theorem 1. It is enough to prove (3), that is,

$$u(T,x) \equiv 0.$$

Suppose that, contrary to this, $u(T,x_0) \neq 0$ for some x_0, $\varphi_1(T) < x_0 < \varphi_2(T)$. Let, for definiteness, $u(T,x_0) = a > 0$ and $x_0 < a$. Let n_0 be such a natural number that $2^{-n_0} < a$. Let $n > n_0$ be an arbitrary natural number. Divide the segment $[a,b]$ of the straight line $t = T$ into n equal parts and, through the division points as well as through the endpoints of the segment, draw straight lines with angular coefficient equal to $(b-a)/n$. Denote by $\Pi_1, \dots, \Pi_n$ the strips between these straight lines, numbered from top to bottom. Let H be the set of values that

takes the value $u(t,x)$ at points where $\operatorname{grad} u = 0$. According to (4), $\mu_1 H = 0$. We may assume that the value α, which the function u takes at the point (T,x_0), does not belong to the exceptional set H (otherwise we could take another point). Therefore, through the point (T,x_0) there passes a smooth component l of the level set. By the maximum principle, l must reach the x-axis while remaining between the curves $x = \varphi_1(t)$ and $x = \varphi_2(t)$. Since the point (x_0,T) is situated to the left of (a,T), l intersects each strip Π_i below the straight line $t = T$.

We may assume that none of the values $\pm 2^{-m}$, $m = 1, 2, \ldots$, belongs to the exceptional set H (multiplying the solution u by a constant, this can always be achieved). Take an arbitrary natural number k, $n \le k \le Mn$, where the integer $M > 2$ will be determined later. Consider the strip Π_i, $i = 1, \ldots, n$, and the intersection of the level set $u = 2^{-k}$ with this strip. Consider those components of the intersection which join the opposite sides of the strip. The set of such components is finite and nonempty (there is at least one component separating, in the strip Π_i, l from the straight line $t = T$). Let $l_{ik}^1, \ldots, l_{ik}^r$ be these components. Next consider the intersection of the level set $u = -2^{-k}$ with the strip Π_i, and also the components of this intersection which join the sides of the strip. Let $\hat{l}_{ik}^1, \ldots, \hat{l}_{ik}^s$ be these components (their set may be empty). Let l_{ik} be the rightmost of all the l_{ik}^p and $\hat{l}_{ik}^q$, $p = 1, \ldots, r$, $q = 1, \ldots, s$. We shall say that the strip Π_i, for the given k, is a strip of the first kind if, in the part G_{ik} of this strip between l_{ik} and the straight line $t = T$, the inequality $u > -2^{-k}$ holds whenever l_{ik} belongs to the level set $u = 2^{-k}$, or $u < 2^{-k}$ whenever l_{ik} belongs to the level set $u = -2^{-k}$. In the contrary case we shall say that the strip Π_i is a strip of the second kind for the given k.

Suppose that for every k there are at least $n/2$ strips which are strips of the first kind. To the domain G_{ik}, lying in a strip Π_i of the first kind between l_{ik} and the straight line $t = T$, Lemma 2 can be applied. Denote by G'_{ik} the set of points $(t, x) \in G_{ik}$ for which the inequality $2^{-(k+1)} < |u(t,x)| < 2^{-k}$ is satisfied. From Lemma 2 it follows that, if for some k the strip Π_i is a strip of the first kind, then $\mu_2 G'_{ik} > \eta(b-a)^3/n^3$. Taking into account that the sets G'_{ik}, $i = 1, \ldots, n$; $k = n, \ldots, Mn$, do not intersect when at least one index differs, from our assumption we find that

$$\mu_2 \bigcup_{i=1}^{n} \bigcup_{k=n}^{Mn} G'_{ik} > \frac{n}{2}(Mn - n)\,\mu_2 G'_{ik} > \frac{M\eta(b-a)^3}{4n}.$$

But the set

$$\bigcup_{i=1}^{n} \bigcup_{k=n}^{Mn} G'_{ik}$$

is situated in the intersection with the domain D of the strip lying between the straight lines passing through the points (a, T) and (b, T) and having angular coefficient $(b-a)/n$. Denote this intersection by E. $\mu_2 E < K/n$, where $K = (b-a)^2\,[\max_t \varphi_2(t) - \min_t \varphi_1(t)]$. Consequently, for every $M > 4K/\eta(b-a)$ and $n > n_0$, among the natural numbers lying between n and Mn one can find a number k for which fewer than $n/2$ strips Π_i will be strips of the second kind. For a strip Π_i of the second kind, in the domain G_{ik} there are points belonging to the level set -2^{-k} if l_{ik} belongs to the level set $+2^{-k}$, and conversely, to the level set $+2^{-k}$ if l_{ik} belongs to the level set -2^{-k}. Since a component of a level set intersecting some strip Π_i must separate all the strips situated below, we may conclude that there are at least $n/2$ curves, situated in D and joining E with the x-axis, not intersecting and ordered from left to right in such a way that on them the function u alternately takes the values $+2^{-k}$ and -2^{-k}. Consequently, in the intersection of D with the strip $0 < t < T-$

$-K/n(b-a) = T_1$, there are no fewer than $n/2$ regions $g_1, \dots, g_s$, $s \geq n/2$, each of which has the following properties: g_i ($i = 1, \dots, s$) has limit points on the lines $t = 0$ and $t = T_1$; on that part of the boundary which is situated strictly inside the strip $0 < t < T_1$, the solution u vanishes; $\max |u(T_1, x)| \geq 2^{-k}$. Among the regions g_i there is a region g_{i_0} such that $\mu_2 g_{i_0} \leq 2\mu_2 D/n$. Applying Lemma 1, we find

$$\max_x |u(0,x)| \geq 2^{\frac{\eta T_1^3 n^2}{4(\mu_2 D)^2}} \cdot 2^{-k},$$

and since for sufficiently large n, $T_1 > T/2$, it follows that, for n greater than some n_1,

$$\max_x |u(0,x)| > 2^{Cn^2} \cdot 2^{-k} > 2^{Cn^2 - Mn},$$

where C is a positive constant depending on the domain and on the equation. The inequality must hold for arbitrary $n > n_1$, which leads us to a contradiction proving the theorem.

Let us note that the conditions $u|_{x=\varphi_1(t)} = u|_{x=\varphi_2(t)} = 0$ were needed in order to prevent the level line from exiting through the lateral boundary of the domain. We would arrive at the same result if we considered a domain on whose lateral boundaries the solution is of constant sign and, in absolute value, exceeds the value on the level line under consideration. This remark makes it possible to apply the preceding arguments (with small modifications) to the proof of Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.