BINARY ADDITIVE PROBLEMS WITH ERGODIC PROPERTIES OF SOLUTIONS

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Abstract

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1. In papers (^{1,2}) an asymptotic formula was found for the number of solutions $Q(n)$ of the Hardy-Littlewood equation:

$$p+\xi^2+\eta^2=n \tag{1}$$

(p is a prime number; ξ, η are integers). For sufficiently large n the formula obtained was:

$$Q(n)=A_0\prod_{p/n}\frac{(p-1)(p-\chi_4(p))}{p^2-p+\chi_4(p)}\frac{1}{\ln n}S_K(n)+R(n), \tag{2}$$

where

$$A_0=\prod_{p>2}\left(1+\frac{\chi_4(p)}{p(p-1)}\right), \quad S_K(n)=\pi n$$

is the area of the circle K of radius $\sqrt{n}$,

$$R(n)=O(n(\ln n)^{-1.042}).$$

In this note we shall generalize equation (1) in a direction which is convenient to interpret from the geometric and ergodic points of view.

Equation (1) can be rewritten in the form

$$p+N(\xi+i\eta)=n, \tag{3}$$

where $\xi+i\eta$ are Gaussian numbers (integer points in the complex plane). Consequently, to each solution of equation (1) we can, on the basis of (3), associate an integer point (ξ, η) of the plane, and this point will lie in the circle K of radius $\sqrt{n}$.

It is natural to pose the following question. Suppose that in the circle K a certain domain Ω is distinguished, increasing together with the circle K. What is the asymptotic formula for the number $Q_\Omega(n)$ of those solutions of equation (1) to each of which there corresponds a point (ξ, η) lying in Ω?

In view of introducing, in the investigation of this problem, elementary ergodic concepts, we shall regard the points (ξ, η) of the solutions of equation (3) as the trajectory of a point moving in the circle K with constant velocity. The question is: how much time will this moving point spend in the given domain Ω?

It is natural to expect that $Q_\Omega(n)$ (or the "time spent by the point in Ω") should be asymptotically proportional to the area S_Ω of the domain Ω.

We introduce a method which makes it possible to answer this question in the case when the area of the domain Ω is not too small in comparison with the area of the circle K.

2. Let first the domain Ω coincide with Δ, where Δ is a circular sector of radius $\sqrt{n}$ with a given aperture angle $\psi = \psi_2 - \psi_1$, $c_1(\ln\ln n)^{-(1/2-\eta_1)} \leq \psi \leq 2\pi$, $c_1 > 0$ is a constant, and η_1 is a small positive number.

Theorem 1. As $n \to \infty$,

$$Q_\Delta(n) = A_0 \prod_{p/n} \frac{(p-1)(p-\chi_4(p))}{p^2 - p + \chi_4(p)} \frac{1}{\ln n} S_\Delta(n)(1+\varepsilon(n)), \tag{4}$$

where

$$\varepsilon(n) = O\left(\frac{1}{\psi_2 - \psi_1}(\ln\ln n)^{-1/2+\eta}\right), \quad 0 < \eta < \min\{1/2, \eta_1\}.$$

A lower estimate for $Q_\Delta(n)$ was obtained by another method by A. A. Polyanskii (^{3}). Theorem 1 can be applied to the study of the behavior of solutions of equation (1) in other domains Ω. In particular, the case is of interest when the star-shaped domain Ω is bounded by a closed smooth contour $C: \; r = r(\varphi),\; 0 \leq \varphi \leq 2\pi,\; \sqrt{n}/(\ln n)^\gamma \leq r \leq \sqrt{n},\; \gamma = 0.04,\; |dr/d\varphi| \leq c_2\sqrt{n}$. Under these conditions the following theorem holds:

Theorem 2. As $n \to \infty$

$$Q_\Omega(n) = A_0 \prod_{p/n} \frac{(p-1)(p-\chi_4(p))}{p^2 - p + \chi_4(p)} \frac{1}{\ln n} S_\Omega(n)(1+\varepsilon(n)), \tag{5}$$

where $\varepsilon(n) = O((\ln\ln n)^{-\eta_2})$, $\eta_2 = -\eta_1 - \eta$.

Theorem 2 is derived from Theorem 1 with the aid of a known device (^{4}).

Corollary of Theorem 2.

$$Q_\Omega(n)/Q(n) \sim S_\Omega(n)/S_K(n) \qquad (n \to \infty). \tag{6}$$

The asymptotic equality (6) follows from (2) and (5) and quite clearly reveals the ergodic character of Theorem 2.

3. We shall set forth the scheme of the proof of Theorem 1.

We shall use the fact that every Gaussian integer $\alpha = \xi + i\eta$ can be represented in the form $\alpha = i^s p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$, where $s = 0, 1, 2, 3$; $p_1, p_2, \dots, p_l$ are prime numbers of the Gaussian field lying in the first quadrant.

Divide the circle K into $K_0 = 2K_1K_2$ equal sectors δ_i $(i = 1, 2, \dots, K_0)$ with opening angle $\varepsilon = \varepsilon_i - \varepsilon_{i-1}$, where $\varepsilon_0 = 0,\ \varepsilon_1 = \varepsilon,\ \varepsilon_2 = 2\varepsilon, \dots, \varepsilon_{K_0} = K_0\varepsilon$; $\varepsilon = 2\pi/K_0$. One may take $K_1 = K_2 = 2^{R-1},\ K_0 = [(\ln\ln n)^{1-\eta}],\ 0 < \eta < 1/2$. Consider the transformation

$$Q(n) = \sum_{p+m=n} r(m), \tag{7}$$

where

$$r(m) = \sum_{\xi^2+\eta^2=m} 1 = 4 \sum_{x/m} \chi_4(x).$$

We divide the set of numbers m satisfying (7) into two classes. In class A we include those m in whose factorization there occurs, at least, one prime number $p_i \equiv 1 \pmod 4$ exactly to the first power, such that $p_i = \mathfrak{p}_i \overline{\mathfrak{p}_i},\ \mathfrak{p}_i \in \delta_i$, for each $i = 1, 2, \dots, K_0/4$ (the δ_i under consideration cover the first quadrant). In class B we include the remaining m. Thus,

$$Q(n) = \sum_{m \in A} \Sigma_A + \sum_{m \in B} \Sigma_B. \tag{8}$$

For estimating the sum Σ_B from above, results from works ($^{3-5}$) connected with sieve methods are invoked. We obtain

$$\Sigma_B = O\left(K_0 \frac{n(\ln\ln n)^3}{(\ln n)(\ln n)^{4/K_0}} \right). \tag{9}$$

For $m \in A$ there already holds an ergodicity of the distribution of integral points on the circle $\xi^2 + \eta^2 = m$.

Indeed, consider K_1 equal sectors of opening $2\varphi = 2K_2\varepsilon$, each of which consists of $2K_2$ consecutive sectors δ_i, beginning with the sector δ_{K_0} (counterclockwise). Taking a fixed number $m \in A$, we can represent it in the form

$$m = \xi^2 + \eta^2 = (\xi + i\eta)(\xi - i\eta) = p_0 p_1 \dots p_{R-2} q_j \overline{p_0}\,\overline{p_1} \dots \overline{p_{R-2} q_j}; \qquad (10)$$

where $N(p_i) \equiv 1 \pmod 4$, there are no equal ones among the p_i, $(N(p_0 p_1 \dots p_{R-2}), N(q_j)) = 1$, and, finally,

$$2^i \varphi - \varepsilon \leq \operatorname{arc} p_i \leq 2^i \varphi,$$

where $i = 0, 1, 2, \dots, R-2$.

From (7) and (10) it follows that

$$r(m) = 2^{R-1} r(q), \qquad \text{where} \qquad q = N(q_j).$$

Therefore all $r(m)$ solutions $\xi + i\eta$ of equation (10) are obtained if one takes any one solution

$$\xi + i\eta = \mathfrak{p}_0 \mathfrak{p}_1 \dots \mathfrak{p}_{R-2} \mathfrak{p}_j$$

and performs the "rotations"

$$\mathfrak{p}_i \to \mathfrak{p}_i, \quad \mathfrak{p}_i \to \overline{\mathfrak{p}}_i, \quad \mathfrak{p}_i \mathfrak{p}_k \to \overline{\mathfrak{p}}_i\,\overline{\mathfrak{p}}_k, \dots, \quad \mathfrak{p}_0 \mathfrak{p}_1 \dots \mathfrak{p}_{R-2} \to \overline{\mathfrak{p}}_0\,\overline{\mathfrak{p}}_1 \dots \overline{\mathfrak{p}}_{R-2}$$

for fixed q_j, and then vary q_j, putting $j = 1, 2, \dots, r(q)$. This gives a set of numbers β of the form

$$\beta = \tilde{\mathfrak{p}}_0 \tilde{\mathfrak{p}}_1 \dots \tilde{\mathfrak{p}}_{R-2}, \qquad \text{where} \qquad \tilde{\mathfrak{p}}_i = \mathfrak{p}_i \text{ or } \overline{\mathfrak{p}}_i.$$

The numbers $\alpha = \beta q_j$ may be regarded as points of a "good trajectory" of the corresponding solution. The "good trajectories" will form the overwhelming majority, since the number of "bad trajectories" for solutions is estimated by formula (9). We have

$$\operatorname{arc} \beta = (\pm 1 \pm 2 \pm 2^2 \pm \dots \pm 2^{R-2}) \varphi + O(R\varepsilon). \qquad (11)$$

The remainder term in (11) will be small in comparison with φ. Numbers of the form $\pm 1 \pm 2 \pm 2^2 \pm \dots \pm 2^{R-2}$ represent all odd numbers from $-(2^{R-1} - 1)$ to $2^{R-1} - 1$, exactly once. Taking into account that $(2^{R-1} - 1)\,\varphi < \pi$, we can number the numbers β in the order of increasing $\operatorname{arc} \beta$. As a result we obtain a sequence of numbers

$$\beta_0, \beta_1, \beta_2, \ldots, \beta_{2^{R-1}-1},$$

distributed uniformly on the circle $\xi^2+\eta^2 = m/q$ (up to small quantities). Hence we are able to derive an estimate for the number $T_\Delta(m)$ of solutions of equation (10) that fall into a given sector Δ. We obtain

$$T_\Delta(m) = \frac{\psi}{2\pi} r(m) \left(1 + O\left(\frac{1}{\psi(\ln\ln n)^{1/2-\eta}}\right)\right), \tag{12}$$

where $m \in A$. Since

$$Q_\Delta(n) = \sum_{p+m=n} T_\Delta(m) = \sum_{p+m=n,\ m\in A} T_\Delta(m) + O\left(\sum_{m\in B} B\right),$$

then, with the help of (1), (8), (9), and (12), we complete the proof of the theorem.

4. In an analogous way one can obtain an asymptotic formula for the generalized Hardy–Littlewood equation

$$p + \varphi(\xi, \eta) = n, \tag{13}$$

where $\varphi(\xi, \eta)$ is a positive quadratic form.

Solutions of equations of the form $a_m + \varphi(\xi, \eta) = n$, where $\{a_m\}$ is a sequence sufficiently well distributed in progressions and $\varphi(\xi, \eta)$ is a positive quadratic form, have ergodic properties similar to those described above.

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Note: Figure translations are in progress. See original paper for figures.

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