ON A MINIMAX PROBLEM IN THE THEORY OF CUBATURE FORMULAS

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Abstract

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ON A MINIMAX PROBLEM IN THE THEORY OF CUBATURE FORMULAS

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Considering a cubature formula on some class of functions Φ, we shall associate with it the error functional

$$(l, \varphi) = \int_\Omega \varphi(x)\, dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}). \tag{1}$$

Here Ω is some domain in the n-dimensional Euclidean space E_n; x is the column vector of coordinates of the variable point. The deepest results in the theory of formulas of mechanical cubature can be obtained under the assumption that the functional class Φ under study forms a Banach space and, moreover, one such that the functional (1), defined on this space, is linear and continuous. In this case it is natural to estimate the quality of a cubature formula by means of the norm of this functional:

$$\|l\|_{\Phi^*} = \sup_{\varphi \in \Phi} \frac{|(l, \varphi)|}{\|\varphi\|}. \tag{2}$$

Finding the minimum of expression (2) with respect to c_k and $x^{(k)}$ is a typical minimax problem. The c_k and $x^{(k)}$ realizing the minimax give the optimal formula. Denote by K the space of infinitely differentiable finite functions in E_n, and by K' the space conjugate to it. Analogously one defines the pair of spaces $K(\Omega)$ and $K'(\Omega)$. We shall also need the class of weight functions B_n. Its elements will be considered continuous in the whole space and satisfying the condition

$$\mu(\xi)\mu^{-1}(\eta) \leq c(1 + |\xi - \eta|^l) \tag{3}$$

for arbitrary ξ and η, with constants c and l depending only on the function $\mu(\xi)$. In what follows the apparatus of Fourier transforms is used, taken in unitary form:

$$\hat{u}(\xi) = \int_{E_n} u(x) e^{2\pi i \xi x}\, dx, \qquad u(x) = \int_{E_n} \hat{u}(\xi) e^{-2\pi i \xi x}\, d\xi. \tag{4}$$

It is known ([3]) that, by weak continuity, Fourier transforms can be transferred to generalized functions.

The space $H_2^{(\mu)}(E_n)$ is defined as the set of those generalized functions $u \in K'$ whose Fourier transforms are square-integrable with weight $\mu^2(\xi)$ ($\mu(\xi) \in B_n$), and the space $H_2^{(\mu)}(\Omega)$ as the set of those generalized functions from $K'(\Omega)$ which are restrictions to the domain $\Omega \subset E_n$ of elements $u \in H_2^{(\mu)}(E_n)$. We introduce the topology in these spaces as follows:

$$\|u\|_{H_2^{(\mu)}(E_n)} = \left(\int |\hat{u}(\xi)|^2 \mu^2(\xi)\, d\xi \right)^{1/2}, \qquad \|u\|_{H_2^{(\mu)}(\Omega)} = \inf \|u^n\|_{H_2^{(\mu)}(E_n)}. \tag{5}$$

where the lower bound in the second of formulas (5) is taken over all such elements $u^n \in H_2^{(\mu)}(E_n)$, whose restrictions to Ω coincide with u, i.e., over all extensions of u to the space E_n. Both spaces are, obviously, isomorphic and isometric to the space L_2. Their deeper properties may be found in ([4]). Let us construct one more space, consisting of periodic generalized functions. A generalized function $u \in K'$ will be called periodic with fundamental period matrix $H(h_1, \dots, h_n)$ (H-periodic) if $u(x + H\beta) = u(x)$, where β is an integer vector. For simplicity we assume that $|H| = 1$. Identifying in the space E_n all points that differ by a period, we obtain an n-dimensional torus Ω_0, over which we shall compute the integral of a periodic function. S. L. Sobolev showed ([1]) that any H-periodic generalized function is representable in the form:

$$u(x) = \dot{u}(x) * \Phi_{0H}(x), \tag{6}$$

where $\dot{u}(x)$ is a function with finite support, $\Phi_{0H}(x) = \sum_\beta \delta(x - H\beta)$.

This makes it possible to obtain, for periodic generalized functions, Fourier-transform formulas analogous to the usual ones:

$$u(x) = \sum_\beta u[\beta] e^{2\pi i \beta H^{-1} x}, \qquad \hat{u}(\xi) = \sum_\beta \hat{u}[\beta] \delta(\xi - \beta H^{-1}), \tag{7}$$

where $\hat{u}[\beta] = \hat{u}[\beta H^{-1}]$. We shall define the space $\widetilde{H}_2^{(\mu)}(\Omega_0)$ as the set of those generalized functions $u \in K'$ for which the sum

$$\|u\|^2_{\widetilde{H}_2^{(\mu)}(\Omega_0)} = \sum_\beta |\hat{u}[\beta]|^2 \mu^2(\beta H^{-1}) \tag{8}$$

is finite.

This space is, obviously, isomorphic and isometric to l_2. In what follows we shall assume that the conditions

$$\int \mu^{-2}(\xi)\, d\xi < \infty, \qquad \sum_{\beta} \mu^{-2}(\beta H^{-1}) < \infty, \tag{9}$$

are satisfied, which guarantees the embedding of the corresponding H-spaces into the space of continuous functions, and consequently also the continuity of the error functional of the cubature formula. Let us first consider the spaces $H_2^{(\mu)}(E_n)$ and $H_2^{(\mu)}(\Omega)$. Introducing in these spaces the scalar product by the formulas

$$(u, v)_{H_2^{(\mu)}(E_n)} = \int \hat{u}(\xi)\, \overline{\hat{v}(\xi)}\, \mu^2(\xi)\, d\xi,$$

$$(u, v)_{H_2^{(\mu)}(\Omega)} = \int \hat{u}^{\,m}(\xi)\, \overline{\hat{v}^{\,m}(\xi)}\, \mu^2(\xi)\, d\xi \tag{10}$$

(where $u^m(x)$ and $v^m(x)$ denote the minimal extensions, respectively, of the elements u and $v \in H_2^{(\mu)}(\Omega)$), we turn them into Hilbert spaces. Therefore one can carry out two realizations of the error functional. On the one hand, starting from the general form of a linear functional in a Hilbert space, we obtain

$$(l_{E_n}, u) = \int \hat{u}_0(\xi)\, \overline{\hat{u}(\xi)}\, \mu^2(\xi)\, d\xi, \qquad (l_{\Omega}, u) = \int \hat{u}_0^m(\xi)\, \overline{\hat{u}^{\,m}(\xi)}\, \mu^2(\xi)\, d\xi, \tag{11}$$

while on the other hand the ordinary realization, accepted in the theory of generalized functions, gives

$$(l_{E_n}, u) = \int l_{E_n}(x) u(x)\, dx, \qquad (l_{\Omega}, u) = \int l_{\Omega}(x) u^m(x)\, dx. \tag{12}$$

Here

$$l_{E_n}(x) = 1 - \sum_{k} c_k \delta(x - x^{(k)}), \qquad l_{\Omega}(x) = \mathcal{E}_{\Omega}(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}). \tag{13}$$

Comparing (11) and (12) with the help of Parseval' s equality, we find

$$\hat{u}_0(\xi) = \hat{l}_{E_n}(\xi)/\mu^2(\xi), \qquad \hat{u}_0^m(\xi) = \hat{l}_{\Omega}(\xi)/\mu^2(\xi), \tag{14}$$

whence, using the theorem on the reciprocity of the Fourier transform for the operations of multiplication and convolution (3), we obtain

$$u_0(x) = \int v'(x-y) l_{E_n}(y)\,dy, \qquad u_0^m(x) = \int v(x-y) l_\Omega(y)\,dy. \tag{15}$$

(By $v(x)$ we denote the Fourier transform of the function $\mu^{-2}(\xi)$.) We shall call these functions extremal. Formulas (15) make it possible to represent the square of the norm of the error functional in the spaces $\widehat{H}_2^{(\mu)}(E_n)$ and $\widehat{H}_2^{(\mu)}(\Omega)$ in the form of a quadratic polynomial in the coefficients c_k. Taking into account that a cubature formula may also be characterized by the relations

$$(l(x), x^{\alpha(j)}) = 0, \qquad |\alpha(j)| < m, \qquad j = 1, 2, \dots, M, \tag{16}$$

we set up, in order to determine the optimal coefficients, a system of Lagrange equations. Analysis of this system makes it possible to establish one important fact showing the connection between the optimality of the coefficients of a cubature formula and the value of the extremal function at the nodes, namely: if the coefficients of the cubature formula are optimal, then the extremal function may be chosen in such a way that its values at the nodes are equal to zero. A statement of this type was first made by the Czech mathematician Ivo Babuška.

The general problem of minimizing the norm of the error functional of a cubature formula, including both the optimal choice of coefficients and of nodes, is very difficult; its complete solution is far from complete even for one-dimensional integrals. In this respect the periodic case is of interest, i.e. the case of the space $\widehat{H}_2^{(\mu)}(\Omega_0)$. From considerations analogous to those given above, using, in addition, the Parseval formula, which here has the form

$$(l(x), u(x)) = \sum_\beta \hat{l}[\beta]\overline{\hat{u}[\beta]}, \tag{17}$$

we find the extremal function

$$u_0(x) = \sum_\beta \frac{l[\beta]}{\mu^2(\beta H^{-1})} e^{2\pi i \beta H^{-1} x}. \tag{18}$$

Let us consider the simplest periodic error functional with fundamental period H,

$$l(x) = 1 - \Phi_{0H}(x) = 1 - \sum_\beta \delta(x - H\beta). \tag{19}$$

Since for (19)

$$\hat{l}[\beta] = 0, \quad \text{if } \beta = 0; \qquad \hat{l}[\beta] = -1, \quad \text{if } \beta \neq 0, \tag{20}$$

for the extremal function we obtain the expression

$$u_0(x) = -\sum_{\beta \neq 0} \frac{1}{\mu^2(\beta H^{-1})} e^{2\pi i \beta H^{-1} x}, \tag{21}$$

and, consequently,

$$\|l\|_{\widetilde{H}_2^{(\mu)*}(\Omega_0)} = (l, u_0)^{1/2} = (u_0(0))^{1/2} = \left(\sum_{\beta \neq 0} \frac{1}{\mu^2(\beta H^{-1})} \right)^{1/2}. \tag{22}$$

Lattices with matrices H and H^{-1} will be called reciprocal. Denoting now the right-hand side of (22) by $B_n^{(\mu)}(H)$, we can formulate the following theorem:

Theorem. Among all periodic error functionals with volume of the fundamental torus equal to one, the smallest norm is possessed by that whose lattice is reciprocal to the lattice realizing the minimum of $B_n^{(\mu)}(H)$.

To make this theorem more concrete and to investigate in greater detail the norm of the error functional, one must know an explicit expression for the weight function. Let us take, for example, $\mu(\xi) = (1 + |\xi|^2)^{1/2}$ (l a positive integer). Then we arrive at the classical space $W_2^{(l)}$ of S. L. Sobolev ([2]). For $n < 2l$ it will be embedded in the space C. Analysis of (22) shows that for large l, $B_n^{(\mu)}(H)$ is minimal when the maximum distance from the origin to the nearest point of the lattice $[H^{-1}]$ is minimal. Consequently, the optimal lattices $[H]$ are those reciprocal to the lattice $[H^{-1}]$ possessing the above-mentioned property (the lattice of the densest packing of balls ([1])).

In a similar way one investigates the spaces of J. Peetre ([5])

$$\left(\mu(\xi) = \left(1 + \sum_{i=1}^{n} \xi_i^2 \right)^s \left(1 + \sum_{i=1}^{n-1} \xi_i^2 \right)^r, \ sr > 0 \right),$$

the spaces of L. N. Slobodetskii ([6])

$$\left(\mu(\xi) = \left(1 + \xi_1^{2l_1} + \dots + \xi_n^{2l_n} \right)^{1/2} \right),$$

where $l_1 \geq 0, \dots, l_n \geq 0$ are arbitrary nonnegative numbers, and a number of other spaces that play an important role in boundary-value problems of mathematical physics.

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Note: Figure translations are in progress. See original paper for figures.