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V. V. RUMYANTSEV

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The basis of the method developed for investigating the stability of stationary motions of holonomic mechanical systems with cyclic coordinates is formed by the theorems of Routh, Poincaré, Kelvin, and Chetaev ($^{1-4}$). The method is characterized by a unified approach to the problem and makes it possible to find, comparatively simply, necessary and sufficient conditions for the "secular" stability (in a certain sense) of stationary motions.

1. Let $q_j, \dot{q}_j$ $(j = 1, \dots, n)$ be independent Lagrangian coordinates and velocities of a system of material points constrained by certain ideal geometric constraints that do not depend explicitly on the time t, and let the coordinates q_α $(\alpha = k+1, \dots, n)$ be cyclic, i.e. $\partial L/\partial q_\alpha = 0$, $L = T - \Pi$ being the Lagrangian function. The equations of motion of the system under the action of potential forces can be written in the form of Routh' s equations

$$\frac{d}{dt}\frac{\partial R_2}{\partial \dot{q}_i} - \frac{\partial R_2}{\partial q_i} = -\frac{\partial W}{\partial q_i} + \sum_{j=1}^{k} g_{ij}\dot{q}_j \quad (i = 1, \dots, k); \tag{1}$$

$$\frac{dq_\alpha}{dt} = -\frac{\partial R}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{dt} = 0 \quad (\alpha = k+1, \dots, n), \tag{2}$$

where the Routh function is $R(q_i, \dot{q}_i, p_\alpha) = R_2 + R_1 - W$. The part of this function that is quadratic with respect to the velocities $\dot{q}_i$ and independent of $\dot{q}_i$,

$$R_2 = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(q_s)\dot{q}_i\dot{q}_j; \qquad W(q_s, p_\alpha) = \Pi(q_s) + \frac{1}{2}\sum_{\alpha,\beta=k+1}^{n} b_{\alpha\beta}(q_s)p_\alpha p_\beta$$

may be interpreted as the kinetic and potential energies of the reduced system (^{5}) with k degrees of freedom, on which there also act gyroscopic forces arising from the linear part R_1 of the Routh function R and characterized by the quantities $g_{ij} = -g_{ji}$. The reduced system is called gyroscopically unconnected if all $g_{ij} = 0$ $(i, j = 1, \dots, k)$. The quadratic form R_2 is a positive definite function of the velocities $\dot{q}_i$. We shall assume that W is an analytic function of the positional coordinates q_i.

Equations (1), (2) admit the energy integral

$$H = \sum_{i=1}^{k} \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} - R = R_2 + W = \text{const}, \tag{3}$$

as well as the cyclic first integrals $p_\alpha = c_\alpha$.

Under certain initial conditions, systems with cyclic coordinates may perform stationary motions in which the positional coordinates and the momenta of the cyclic coordinates retain their initial values $q_i = q_i^0$, $\dot{q}_i = 0$, $p_\alpha = p_\alpha^0$, while the cyclic coordinates vary linearly with time. For fixed values $p_\alpha = c_\alpha$, the constants q_i^0 are determined from the equilibrium equations of the reduced system

$$\partial W / \partial q_i = 0 \quad (i = 1, \dots, k). \tag{4}$$

The general theory of equilibrium of systems whose potential energy depends not only on the coordinates but also on certain parameters was proposed by Poincaré ([2,4]). Let

$$q_i = \varphi_i^{(s)}(c_{k+1}, \dots, c_n) \tag{5}$$

be the roots of equations (4). We shall assume that the functions $\varphi_i^{(s)}(c_\alpha)$ are continuous functions of the parameters c_α. Consequently, the stationary motions form manifolds of dimension $n - k$, each point of which is a stationary point of the function W for fixed c_α. The roots of equations (4) are determined uniquely at all ordinary points of the n-dimensional space G of the variables q_i, c_α, where $\Delta = \|\partial^2 W / \partial q_i \partial q_j\| \neq 0$. Points at which $\Delta = 0$ are called critical points, or bifurcation points. At such points the separate surfaces (5), which together constitute a certain real surface B of stationary motions, may intersect one another.

2. Let us consider the question of Lyapunov stability of a certain stationary motion with respect to q_i, $\dot{q}_i$, p_α. Without loss of generality one may assume that, for the given fixed values of the parameters $p_\alpha = c_\alpha$, the stationary motion corresponds to the values $q_i = 0$. The equations of the perturbed motion will be equations (1), (2), in which we put $p_\alpha = c_\alpha + \eta_\alpha$. From the latter there follows at once stability with respect to p_α and instability with respect to q_α.

The well-known theorem of Routh (^{1,4}) is:

Theorem 1. *If the function $V = H - H_0$ is a sign-definite function of the variables q_i, $\dot{q}_i$, then the stationary motion is stable with respect to these variables, provided that the values of the constants c_α are not perturbed.*

Here H_0 denotes the function H when in the latter all positional coordinates and their velocities are set equal to zero. We note that, since the function R_2 is positive definite with respect to q_i, the requirements of Theorem 1 reduce to the requirement of positive definiteness of the function $W - W_0$, as a consequence of which Routh' s theorem may be formulated as Lagrange' s theorem (^{4}) for the reduced system.

Consequently, stationary motions are stable for all points of an isolated (for fixed c_α) minimum of the function W.

Following Routh (^{1}), one may also give the following formulation of the theorem on stability of stationary motions (^{6}):

If, for some system, a certain number of integrals independent of time has been found, and if among these there exists one which can have a minimum or a maximum, for given values of the remaining integrals, for certain definite values of the variables entering it, then these values will in general correspond to one of the actual motions of the system, which will be stable with respect to these variables at least for perturbations that do not change the values of the remaining integrals.

Lyapunov (^{6}) gave the following addition to Routh' s theorem:

If the integral under consideration has a minimum or a maximum also for all values sufficiently close to the given values of the remaining integrals, and if the values of the variables which bring it to a minimum or maximum are continuous functions of the values of these integrals, then the motion under consideration will be stable for all perturbations.

From the geometrical point of view this theorem is almost obvious. Indeed, if the unperturbed motion corresponds to some ordinary point of the space G, for which the function W has (for the given c_α) an isolated minimum, then, for sufficiently small perturbations $\eta_\alpha \neq 0$, the perturbed motion remains at all times in a neighborhood of the minimum point of W corresponding to the values $p_\alpha = c_\alpha + \eta_\alpha$.

It is known that a stationary motion can also be stable in the case when the function W, for the given c_α, has no minimum; in such cases gyroscopic stabilization takes place (^{4}). Thus, the converse of Routh' s theorem, when the function W has no minimum, can hold only under additional conditions. The simplest case of the converse of Routh' s theorem is represented by gyroscopically unconnected systems, for which, obviously, the converse of Lagrange' s theorem, given by N. G. Chetaev ((^{4}), p. 236), is applicable. Consequently, the following is true.

Theorem 2. *If, for an isolated (with fixed p_α) stationary motion of a gyroscopically unconnected system, the function W, assumed to be an analytic function of q_i, has no minimum, then the stationary motion is unstable.*

For a gyroscopically connected system, in some cases the following may prove useful.

Theorem 3. *If the function*

$$2R_2 + R_1 + \sum_{i=1}^{k} q_i \frac{\partial R}{\partial q_i}$$

is positive definite with respect to q_i, $\dot{q}_i$ in the region

$$(C): \; H - H_0 < 0, \quad \sum_{i=1}^{k} q_i \frac{\partial R}{\partial q_i} > 0,$$

then the stationary motion $q_i = 0$ is unstable.

Corollary. *If in the region (C) the quadratic part $R^{(2)}$ of the expansion of Routh' s function R in a Maclaurin series is positive definite, then the stationary motion is unstable.*

Consider the secular equation

$$\Delta(\lambda) = \left\| \left(\partial^2 W / \partial q_i \partial q_j\right)_0 - \lambda \delta_{ij} \right\| = 0, \qquad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{6}$$

Kelvin' s theorems (^{3,4}) on the possibility or impossibility of gyroscopic stabilization of an isolated equilibrium position of a system, depending on whether the number of negative roots of equation (6) is even or odd, are well known. Consequently, the following is true.

Theorem 4. *If the number of negative roots of equation (6) is odd and there is not a single root equal to zero, then the stationary motion is unstable; if the number of negative roots is even, then under certain conditions gyroscopic stabilization is possible.*

3. Let us determine the influence of dissipative forces on the stability of stationary motions in the case when the Rayleigh function

$$2f(q_s, \dot{q}_s) = \sum_{i,j=1}^{k} e_{ij}(q_s) \dot{q}_i \dot{q}_j \tag{7}$$

is a negative definite function of the velocities of all positional coordinates. The equations of the perturbed motion in this case still have the first integrals $\eta_\alpha =$

const. Instead of the energy integral, we obtain an equation for the rate of dissipation of energy,

$$\frac{d}{dt}(R_2 + W) = 2f. \tag{8}$$

Using this equation, one can prove the following theorems.

Theorem 5. *If the function W has an isolated minimum for the given quantities $p_\alpha = c_\alpha$, and also for all sufficiently close to the given $p_\alpha = c_\alpha + \eta_\alpha$, and if the values q_i that bring it to a minimum are*

continuous functions of the quantities p_α, then the dissipative forces do not destroy the stability of the stationary motion corresponding to the given $p_\alpha = c_\alpha$, and every perturbed motion sufficiently close to it tends asymptotically to the stationary motion corresponding to the values of the quantities $p_\alpha = c_\alpha + \eta_\alpha$.

Corollary. If the function W has an isolated minimum for the given $p_\alpha = c_\alpha$, then the stationary motion becomes asymptotically stable upon the addition of dissipative forces, provided the values of the constants c_α are not perturbed ([3,4]).

Theorem 6. If, for an isolated stationary motion corresponding to the given c_α, the function W has no minimum and, in an arbitrarily small neighborhood of this motion, can assume negative values, then the stationary motion is unstable ([7]).

Corollary. The gyroscopic stabilization of a stationary motion, possible in the case where W has no minimum and the equation (6) has an even number of negative roots, is destroyed under the action of dissipative forces.

Remark. If the absence of a minimum of the function W for an isolated stationary motion can already be detected from its second variation $W^{(2)}$, then Theorem 6 becomes Kelvin' s theorem ([3,4]) on the impossibility of gyroscopic stabilization of the equilibrium position of a system under the action of dissipative forces with complete dissipation.

Following Kelvin ([3,4]), one may thus distinguish between temporary and "secular" stability of stationary motions. It should only be emphasized that, when speaking in this sense of the "secular" stability of stationary motions, we assume that the requirement of constancy of the momenta $p_\alpha = c_\alpha + \eta_\alpha$ is satisfied throughout the entire motion.

It follows from Theorems 5 and 6 that, for the "secular" stability of a stationary motion (under the condition of constancy of the cyclic momenta p_α), it is necessary (in a certain sense) and sufficient that, for all given $p_\alpha = c_\alpha + \eta_\alpha$, $|\eta_\alpha| \leq \delta$, the function W have an isolated minimum.

We note that a change of stability of stationary motions can occur only at critical points. For stationary motions there is also a law of change of "secular" stability at fixed values of the parameters c_α (^{2,4}).

Computing Center
Academy of Sciences of the USSR

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