ON THE QUESTION OF ISOMORPHISMS OF AN ANALYTIC SPACE COMMUTING WITH A POWER OF THE DIFFERENTIATION OPERATOR

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83+517.948.35+517.948.5

MATHEMATICS

N. I. NAGNIBIDA

ON THE QUESTION OF ISOMORPHISMS OF AN ANALYTIC SPACE COMMUTING WITH A POWER OF THE DIFFERENTIATION OPERATOR

(Presented by Academician A. N. Kolmogorov on 26 VIII 1965)

The problem of finding the general form of isomorphisms Y of the space $\mathfrak{A}_\infty$ of entire functions that commute with the operator $D^n = d^n/dz^n$, $n \geqslant 1$, was posed by Delsarte and Lions in (^{1}). According to Theorem 2.1 of (^{1}), every such isomorphism Y satisfies, in the case $n \geqslant 2$, the relation

$$Y \exp(\lambda z) = \sum_{j=0}^{n-1} M_j(\lambda) \exp(\lambda z \omega^j), \qquad \omega = \exp \frac{2\pi i}{n}, \tag{1}$$

where $M_j(\lambda)$, $0 \leqslant j \leqslant n-1$, are entire functions of exponential type with determinant

$$\Delta(\lambda) = \det \left\| \begin{matrix} M_0(\lambda) & M_1(\lambda) & \cdots & M_{n-1}(\lambda) \\ M_{n-1}(\lambda\omega) & M_0(\lambda\omega) & \cdots & M_{n-2}(\lambda\omega) \\ \cdots & \cdots & \cdots & \cdots \\ M_1(\lambda\omega^{n-1}) & M_2(\lambda\omega^{n-1}) & \cdots & M_0(\lambda\omega^{n-1}) \end{matrix} \right\| \equiv \text{const} \neq 0. \tag{2}$$

Recently the result of Delsarte and Lions was carried over by I. Ya. Viner (^{2}) to the spaces $\mathfrak{A}_R$, $0 < R < \infty$, of all single-valued analytic functions in the disk $|z| < R$ with the topology of compact convergence (^{3}). It turns out, however, that the results of Delsarte-Lions and Viner are erroneous, i.e., the full group of isomorphisms of the space $\mathfrak{A}_R$ (respectively, $\mathfrak{A}_\infty$) commuting with D^n, $n \geqslant 2$, is not described by conditions (1), (2). Indeed, denoting by A_0 and A_1 the linear continuous operators in $\mathfrak{A}_R$ ($\mathfrak{A}_\infty$) defined by the relations

$$A_0 f(z) = \frac{f(z) + f(-z)}{2}, \qquad A_1 f(z) = \frac{f(z) - f(-z)}{2},$$

let us consider the operators

$$Y = (I + E)A_0 + (D^3 + D^2 - D)A_1,$$

$$Y_1 = (D^2 - I)A_0 + (E + D - D^3)A_1;$$

where

$$If(z) = \int_0^z f(\zeta)\,d\zeta$$

and $Ef(z) = f(z)$. Since (as is easily verified on the basis elements z^k, $k = 0, 1, \ldots$) $YD^2 = D^2Y$ and $YY_1 = Y_1Y = E$, it follows that Y is an isomorphism of the space $\mathfrak{U}_R$ ($\mathfrak{U}_\infty$) commuting with D^2. Further, we have $Y1 = 1 + z$, although if the theorems of Delsarte-Lions and Viner were valid, then from (1), as $\lambda \to 0$, it would follow that $Y1 = \text{const} \neq 0$. Moreover, writing relation (1) for this operator, we easily obtain

$$M_0(\lambda) = \frac{1}{2}\left(\frac{1}{\lambda} + 1 - \lambda + \lambda^2 + \lambda^3\right),$$

$$M_1(\lambda) = \frac{1}{2}\left(-\frac{1}{\lambda} + 1 - \lambda - \lambda^2 + \lambda^3\right), \qquad \Delta(\lambda) \equiv 1.$$

Thus, the operator Y constructed does not satisfy either of the conditions (1), (2).

The present paper is devoted to finding the general form of the isomorphisms of the spaces $\mathfrak{U}_R$ and $\mathfrak{U}_\infty$ considered here, relying on the matrix representation of linear continuous operators in these spaces (^{4,5}).

1. If T is a linear operator in $\mathfrak{U}_R$ ($\mathfrak{U}_\infty$), commuting with D^n, $n \geq 1$, then the elements of its matrix $\{t_{i,k}\}_{i,k=0}^{\infty}$ in the power basis $\{z^k\}_{k=0}^{\infty}$, i.e.

$$Tz^k = \sum_{i=0}^{\infty} t_{i,k} z^i, \qquad k = 0, 1, \ldots,$$

are necessarily connected by the relations

$$t_{sn+p,\,mn+q} = \begin{cases} 0, & m < s, \\ \dfrac{(mn+q)!\,p!}{(sn+p)!\,[(m-s)n+q]!}\,t_{p,\,(m-s)n+q}, & 0 \leq s \leq m < \infty; \\ & 0 \leq p,\ q \leq n-1. \end{cases} \tag{3}$$

Let us introduce the operators A_q, $0 \le q \le n-1$, by putting

$$A_q f(z) = A_q\left(\sum_{i=0}^{\infty} a_i z^i\right) = \sum_{s=0}^{\infty} a_{sn+q} z^{sn+q},$$

where $f(z)$ is an arbitrary function from $\mathfrak{A}_R$ $(\mathfrak{A}_\infty)$.

From the relations (3) and the description of linear continuous operators in $\mathfrak{A}_R$ $(\mathfrak{A}_\infty)$ (^{4,5}) one may obtain

Theorem 1. In order that a linear operator T be a continuous operator in the space $\mathfrak{A}_R$ $(\mathfrak{A}_\infty)$, commuting with D^n, $n \ge 1$, it is necessary and sufficient that it have the form

$$T = \sum_{s=0}^{\infty}\sum_{q=0}^{n-1}\sum_{p=0}^{n-1} t_{p,\,sn+q}\frac{p!}{(sn+q)!}D^{sn+q-p}A_q^{*} \tag{4}$$

and satisfy the condition: for every $\rho < R$ $(\rho < +\infty)$ there exists an $r = r(\rho) < R$ $(r < +\infty)$ such that

$$\sup_{0 \le mn+q < \infty}\sum_{s=0}^{m}\sum_{p=0}^{n-1}\frac{(mn+q)!\,p!}{(sn+q)!\,[(m-s)n+p]!}\left|t_{p,\,sn+q}\right|\frac{\rho^{(m-s)n+p}}{r^{mn+q}} < +\infty. \tag{5}$$

Theorem 2. For condition (5) to hold in the space $\mathfrak{A}_R$ $(\mathfrak{A}_\infty)$, it is necessary and sufficient that for each (some) ε, $\varepsilon > 0$, there exist a constant $M(\varepsilon) > 0$ such that

$$|t_{p,j}| \le M(\varepsilon)\varepsilon^j, \qquad j \ge 0; \quad 0 \le p \le n-1. \tag{6}$$

Corollary. In order that a linear operator T be a continuous operator in $\mathfrak{A}_R$ $(\mathfrak{A}_\infty)$, commuting with D^n, $n \ge 1$, it is necessary and sufficient that it have the form (4) and satisfy the corresponding condition (6).

Finally, note that when condition (6) is fulfilled, the functions

$$\psi_{p,q}(\lambda) = \sum_{m=0}^{\infty} t_{p,\,mn+q}\frac{\lambda^{mn}}{(mn+q)!}, \qquad 0 \le p,\ q \le n-1,$$

are entire of class not exceeding $[1,0]$ (respectively, functions of finite degree).

---

* In this formula, for $s = 0$ and $q < p$, $\quad D^{q-p} = I^{p-q}$.

2. Let us proceed to find all isomorphisms T of the space $\mathfrak{A}_R$ $(\mathfrak{A}_\infty)$ that commute with D^n, $n \geqslant 2$. Setting

$$\varphi_q(\lambda, z) = \sum_{m=0}^{\infty}\frac{\lambda^{mn}z^{mn+q}}{(mn+q)!}, \qquad 0 \leqslant q \leqslant n-1,$$

we obtain

$$T\varphi_q(\lambda, z) = \sum_{p=0}^{n-1} p!\psi_{p,q}(\lambda)\varphi_p(\lambda, z), \qquad 0 \leqslant q \leqslant n-1.$$

Proceeding from these relations and from the consequence of Theorem 2, it is easy to prove the following theorem.

Theorem 3. *In order that a linear operator T be an isomorphism of the space $\mathfrak{A}_R$ ($\mathfrak{A}_\infty$) that commutes with D^n, $n \geqslant 2$, it is necessary and sufficient that it have the form (4), satisfy the corresponding condition (6), and*

$$\det \|\psi_{p,q}(\lambda)\|_{p,q=0}^{n-1} \equiv \text{const} \neq 0.$$

Hence, in particular, it follows that every isomorphism of the space $\mathfrak{A}_R$ that commutes with D^n, $n \geqslant 2$, is at the same time an isomorphism of $\mathfrak{A}_\infty$.

3. Now let some isomorphism T of the space $\mathfrak{A}_R$ ($\mathfrak{A}_\infty$), commuting with D^n, $n \geqslant 2$, satisfy conditions (1), (2). Then from (1) and (4) it follows that

$$\sum_{j=0}^{n-1} M_j(\lambda)\lambda^p\omega^{pj} = \sum_{q=0}^{n-1} p!\lambda^q\psi_{p,q}(\lambda), \qquad 0 \leqslant p \leqslant n-1.$$

Therefore, taking into account relations (3), we must conclude that the following theorem is true.

Theorem 4. *In order that an isomorphism of the space $\mathfrak{A}_R$ ($\mathfrak{A}_\infty$), commuting with D^n, $n \geqslant 2$, satisfy conditions (1), (2), it is necessary that its matrix have upper triangular form.*

4. Let us consider one application of the results presented above. Let

$$A = \sum_{j=0}^{n} a_j(z)D^j,$$

where $a_j(z) \in \mathfrak{A}_R(\mathfrak{A}_\infty)$ and $a_n(z) \equiv 1$. As is known (^{1,6,7}), the operators A and D^n are equivalent to one another, i.e. there exists such an isomorphism T_0 of the space $\mathfrak{A}_R$ ($\mathfrak{A}_\infty$) (in this case there are infinitely many of them) that $T_0D^n = AT_0$. In the works (^{1,6,7}) it is indicated that the operator T_0 can be chosen so that

$$D^\nu T_0 f(z)\big|_{z=0} = D^\nu f(z)\big|_{z=0}, \qquad 0 \leqslant \nu \leqslant n-1,$$

and all other transformation operators T_1, $T_1 D^n = A T_1$, can be obtained by the formula $T_1 = T_0 T$, where T ranges over all possible isomorphisms of $\mathfrak{U}_R(\mathfrak{U}_\infty)$ that commute with D^n.

Let C be an operator in the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$, determined by the matrix $\{c_{i,k}\}_{i,k=0}^{\infty}$; $c_{i,k} = 0$, if $\min(i,k) \geqslant n$.

Theorem 5. *In order that there exist an isomorphism T_1 of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$ such that*

$$T_1 D^n = A T_1,$$

$$D^\nu T_1 f(z)\big|_{z=0} = D^\nu C f(z)\big|_{z=0}, \qquad 0 \leqslant \nu \leqslant n-1; \qquad n \geqslant 2, \tag{7}$$

it is necessary and sufficient that

$$\det \|c_{i,k}\|_{i,k=0}^{n-1} \neq 0. \tag{8}$$

Under these conditions the operator T_1 is determined uniquely.

Remark. If

$$B = \sum_{j=0}^{n} b_j(z) D^j, \quad b_j(z) \in \mathfrak{U}_R\ (\mathfrak{U}_\infty), \quad b_n(z) \equiv 1$$

and (8) is satisfied, then one can also assert the existence of such an isomorphism T_2 of the space $\mathfrak{U}_R$ $(\mathfrak{U}_\infty)$ which satisfies relations (7) and $T_2 B = A T_2$.

In conclusion I express my sincere gratitude to K. M. Fishman for useful advice.

Chernivtsi State
University

Received
28 IV 1965

REFERENCES

1. J. Delsarte, J. L. Lions, *Comm. Math. Helv.*, **32**, No. 2, 113 (1957).
2. I. Ya. Viner, *UMN*, **20**, issue 1 (121), 185 (1965).
3. G. Köthe, *J. reine u. angew. Math.*, **191**, 30 (1953).
4. M. G. Khaplanov, *DAN*, **80**, No. 1, 21 (1951).

5. K. M. Fishman, *DAN*, **127**, No. 1, 40 (1959).

6. M. K. Fage, in: *Collected Papers: Studies on Modern Problems of the Theory of Functions of a Complex Variable*, Moscow, 1961, p. 468.

7. K. M. Fishman, *UMN*, **19**, issue 5 (119), 143 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.