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Abstract

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ON THE FLUTE INSTABILITY OF PLASMA IN TOROIDAL GEOMETRY

In the present work we consider the question of the flute instability of an ideally conducting plasma in toroidal discharges with a strong longitudinal magnetic field, i.e., in systems of the "Tokamak"type ([1]). Usually, as the stability criterion for an ideal plasma in such systems, the Suydam condition ([2]), obtained for a straight column, is adopted. This condition has the form

$$-8\pi p' < \frac{rH_z^2}{4}\left(\frac{q'}{q}\right)^2, \tag{1}$$

where p is the plasma pressure; H_z is the longitudinal magnetic field; $q = \dfrac{rH_z}{R_0 H_\varphi}$, the so-called safety factor with respect to helical instability; H_φ is the azimuthal magnetic field; r is the minor radius; R_0 is the major radius of the torus; the prime denotes differentiation with respect to r.

However, for Tokamak-type systems with a strong longitudinal magnetic field, i.e., for $H_z \gg H_\varphi$, condition (1) is not applicable. Indeed, the flute instability ultimately arises because of the curvature of the lines of force. In a straight discharge the radius of curvature is $R_s = rH_z^2/H_\varphi^2$, and for $H_z \gg H_\varphi$ it considerably exceeds r. It is clear that when $R_s > R_0$ the curvature of the lines of force due to toroidality begins to play an essential role. This curvature changes sign when the minor azimuth φ changes: on the outer contour of the torus the lines of force are convex, while on the inner contour they are concave. Therefore, for $p' < 0$, only the outer region leads to instability, and the instability itself takes the form of the so-called ballooning mode ([3,4]), in which the perturbation of the lines of force on the favorable inner contour is small, i.e., the corresponding ends of the lines of force are as though fixed. For $p' > 0$, the instability must develop on the inner contour of the torus.

As is known ([2,5]), within the framework of ideal magnetohydrodynamics, instability sets in when, along with the initial axisymmetric equilibrium state, the possibility appears of forming an equilibrium configuration with a perturbed magnetic field. Therefore, to find the stability criterion it is sufficient to investigate the equilibrium equation

$$\nabla p = \frac{1}{c}[\mathbf{j}\mathbf{H}] = \frac{1}{4\pi}[\mathrm{rot}\,\mathbf{H}\,\mathbf{H}], \tag{2}$$

and to determine under what conditions an equilibrium close to the axisymmetric one exists.

In the case considered by us, $H_z \gg H_\varphi$, instead of (2) it is more convenient to use the relations following from (2),

$$\mathbf{H}\nabla p = 0, \qquad \mathbf{j}_\perp = c[\mathbf{H}\nabla p]/H^2, \tag{3}$$

where $\mathbf{j}_\perp$ is the component of the current density transverse with respect to $\mathbf{H}$, so that the complete vector $\mathbf{j}$ can be represented in the form

$$\mathbf{j} = \mathbf{j}_\perp + \alpha\mathbf{H}, \qquad \alpha = \frac{c}{4\pi H^2}\,\mathbf{H}\,\mathrm{rot}\,\mathbf{H}. \tag{4}$$

From the condition $\mathrm{div}\,\mathbf{j} = 0$, taking into account $\mathrm{div}\,\mathbf{H} = 0$ and (2), (3), (4), we obtain

$$\mathbf{H}\nabla\alpha + \mathrm{div}\,\mathbf{j}_\perp = \mathbf{H}\nabla\alpha + \frac{2[\mathbf{H}\nabla\mathbf{H}]}{H^3}\nabla p = 0. \tag{5}$$

Let us now consider some equilibrium axisymmetric state p^0, H_z^0, H_φ^0 and linearize equations (3), (4), (5), assuming that the perturbation p', H_z', H_φ' is small. In doing so we shall take into account that $H_\varphi^0 \ll H_z^0,\ p^0 \ll H_\varphi^{0\,2}/8\pi$.

Since H_z is large, the perturbation H_z' may be neglected (otherwise a very large perturbation of the magnetic-field pressure $H_z' H_z^0/4\pi$ would arise). For the same reason one may neglect the perturbation of H, and by H one may understand H_z^0. For small toroidality, i.e. $r/R_0 \ll 1$, which we shall assume to hold, the longitudinal magnetic field, decreasing as $1/R$ with distance from the axis of symmetry of the torus, may be written in the form

$$H_z^0 \simeq H_0\left(1 - \frac{r}{R_0}\cos\varphi\right),$$

where H_0 is the longitudinal magnetic field on the magnetic axis, and φ is the small azimuthal angle. Thus, in equation (5), by ∇H one may understand

$$H_0\nabla\frac{r}{R_0}\cos\varphi,$$

and H^3 may be approximately replaced by H_0^3. It is precisely the second term in (5) that takes account of the effect of the curvature of the lines of force in toroidal geometry. In all the remaining expressions the toroidality enters as a small correction to the principal terms, and it may be neglected, assuming that we are dealing with a straight cylindrical column of finite length. We shall consider the perturbation of this column in the cylindrical coordinate system r, φ, z.

Taking the above remarks into account, the linearized equation (5) takes the form

$$\mathbf{H}^0 \nabla \alpha' + H_r' \frac{d\alpha_0}{dr} - \frac{2[\mathbf{H}^0 \mathbf{e}_x]}{H_0^2 R_0} \nabla p' - \frac{2[\mathbf{H}' \mathbf{e}_x]}{H_0^2 R_0} \nabla p_0 = 0, \tag{6}$$

where $\mathbf{e}_x = \nabla r \cos\varphi$ is the unit vector directed along the x-axis. In the cylindrical coordinate system r, φ, z, the vector $\mathbf{e}_x = \{\cos\varphi, -\sin\varphi, 0\}$. Since ∇p_0 is directed along the radius r, the last term in (6) is proportional to H_z' and is therefore negligibly small. In view of $H_z \gg H_\varphi$, the quantity α_0 is equal to

$$\alpha_0 \simeq \frac{1}{H_0} j_0 = \frac{c}{4\pi H_0} \frac{1}{r} \frac{d}{dr} r H_\varphi^0, \tag{7}$$

and the perturbation α' is equal to

$$\alpha' \simeq \frac{c}{4\pi} \frac{\mathbf{H}_0}{H_0^2} \operatorname{rot} \mathbf{H}' = \frac{c}{4\pi H_0} \left\{ \frac{1}{r} \frac{\partial}{\partial r} r H_\varphi' - \frac{1}{r} \frac{\partial}{\partial \varphi} H_r' \right\}. \tag{8}$$

Since H_z' is small, the continuity equation $\operatorname{div} \mathbf{H}' = 0$ takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} r H_r' + \frac{1}{r} \frac{\partial}{\partial \varphi} H_\varphi' = 0. \tag{9}$$

It follows from this that the perturbation of the magnetic field can be expressed in terms of the current function ψ,

$$r H_r' = \partial\psi / \partial\varphi, \qquad H_\varphi' = -\partial\psi / \partial r. \tag{10}$$

Substituting these expressions into (8), and then α' into (6), we obtain

$$\mathbf{H}^0 \nabla \Delta_\perp \psi + \frac{4\pi}{cr} \frac{dj_0}{dr} \frac{\partial\psi}{\partial\varphi} - \frac{8\pi}{cR_0} \left\{ \sin\varphi \frac{\partial p'}{\partial r} + \cos\varphi \frac{1}{r} \frac{\partial p'}{\partial\varphi} - b \right\}, \tag{11}$$

where

$$\Delta_\perp \psi = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial\psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2},$$

and the term $b \sim r^2 R_0^2$ takes account of the contribution from the third term in (6).

To find b, we take into account that the flute instability develops on perturbations strongly elongated along the field lines, when $\mathbf{H}^0 \nabla a' \to 0$, i.e., it is a slowly varying function of φ along the field line. Consequently, averaging over φ must be performed in the sense of averaging along the field line with weight $1/H_0$. The corresponding calculations of (6) show that

$$b = \frac{\mathbf{H}_0}{H_0^2 U} [\nabla p' \nabla U], \quad \text{where } U = U(r) = \int \frac{dl}{H_0},$$

and dl is an element of length along the field line. Adding here the linearized equation (3)

$$\mathbf{H}^0 \nabla p' + \frac{1}{r} \frac{dp^0}{dr} \frac{\partial \psi}{\partial \varphi} = 0, \tag{12}$$

we obtain a system of two equations for p' and ψ.

Let us note that in the case of a straight pinch ($R_0 \to \infty$) for $H_\varphi^0 \ll H_z^0$, taking into account small quantities of order H_φ^2 / H_z^2, we would obtain an equation of the form (11), in which instead of the last two terms there would stand the expression

$$\frac{4\pi}{cR_s} \frac{1}{r} \frac{\partial p'}{\partial \varphi} \equiv \frac{4\pi H_\varphi^2}{cH_z^2} \frac{1}{r^2} \frac{\partial p'}{\partial \varphi}.$$

It is seen from this that toroidality begins to play a role when $R_0 < R_s$. Owing to periodicity in φ and z, an arbitrary solution of the system of equations (11), (12) may be represented in the form $\psi = \psi_{mn}(\varphi, r) \exp(im\varphi - inz/R_0)$, and analogously for p', where the functions $\psi_{mn}(\varphi, r)$, $p_{mn}(\varphi, r)$ may be assumed to have a minimum number of nodes in φ. In other words, if ψ_{mn} and p_{mn} are expanded in a Fourier series in φ, i.e. $\psi_{mn} = \sum_l \psi_l \exp(il\varphi)$, then ψ_l must decrease with the number l. Substituting this expansion into (11), (12) and passing from the current function ψ to the quantity

$$\xi_l = \psi_l / k_l, \quad \text{where } k_l = \frac{m + l}{r} \frac{H_\varphi^0}{H_z^0} - \frac{n}{R_0},$$

after eliminating p_{mn} we obtain the system of equations for the harmonics ξ_l:

$$k_l\Delta_\perp k_l\xi_l + \frac{4\pi(m+l)}{crH_0}\frac{dj_0}{dr}k_l\xi_l - A\Big\{-2\frac{R_0}{R_s}(m+l)^2\xi_l + (m+l-1)^2\xi_{l-1}$$
$$+ (m+l+1)^2\xi_{l+1} + r\frac{\partial}{\partial r}[(m+l-1)\xi_{l-1} - (m+l+1)\xi_{l+1}] - \frac{2U'R_0}{U}(m+l)^2\xi_l\Big\} = 0. \quad (13)$$

Here the notation

$$A = -\frac{4\pi}{R_0 r H^2}\frac{dp^0}{dr}$$

has been introduced, and for completeness a small term $\sim 1/R_s^2$ has been included. We are interested in local perturbations of the flute type developing near the singular point $r = r_0$, where the longitudinal wave number

$$k_0 = \frac{m}{r}\frac{H_\varphi^0}{H_z^0} - \frac{n}{R_0}$$

vanishes. In the neighborhood of this point, k_0 may be expanded in $x = r - r_0$:

$$k_0 \simeq \frac{m}{r^2}\theta x, \quad \text{where } \theta = \frac{r^2 q'}{R_0 q^2}, \quad q' = \frac{dq}{dr}.$$

Moreover, in (13) the second term may be neglected in comparison with the last. For the local-type perturbations of interest to us, $m \gg 1$. The localization width of such perturbations is $\Delta x \sim r/m$. As we shall see below, it suffices for us to consider the case $rq'/q \ll 1$. In this case the singular point for perturbations with $l = 1$, determined by the relation

$$k_1 = \frac{m+1}{r}\frac{H_\varphi^0}{H_z^0} - \frac{n}{R} \simeq \left(\frac{mq'}{2q}x + 1\right)\frac{1}{r}\frac{H_\varphi^0}{H_z^0} = 0,$$

lies far outside the localization region of the perturbation and, consequently, k_1 may be regarded as constant and equal to $H_\varphi/H_z r$. Estimates show that, for $p_\varphi \equiv 8\pi p/H_\varphi^2 \ll R_0/r$, that

is practically always satisfied, the harmonics ξ_l rapidly decrease with the number l. Therefore, in the system of equations (13) it is sufficient to restrict oneself to only two harmonics. Acting with $\Delta_\perp$ on equation (11) for the zeroth harmonic and then eliminating ψ_1, ψ_{-1} with the aid of equations (11) for the first harmonics, we obtain for ψ_0 the equation

$$\Delta_\perp x \cdot \Delta_\perp x \cdot \psi_0 + \gamma \Delta_\perp \psi_0 = \delta \left(\frac{m^2}{r^2} \psi_0 - \frac{\partial^2}{\partial x^2} \psi_0 \right), \tag{14}$$

where

$$\delta = \frac{1}{2} \left(\frac{r}{R_0} \frac{8\pi p'}{H_\varphi^2} \right)^2 \left(\frac{q}{q'} \right)^2, \qquad \gamma = -\frac{8\pi p'}{r H_z^2} \left(\frac{q}{q'} \right)^2 - \frac{\pi p' U'}{H_\varphi^2 U} \left(\frac{q}{q'} \right)^2.$$

To solve this equation it is convenient to pass to the Fourier transform with respect to x: $\psi_0(x) = \int \psi_0(k) \exp(ikx)\, dk$. In the Fourier representation, equation (14) is reduced to the form

$$\frac{d^2 u}{d\chi^2} - \frac{1 - (\gamma + \delta)(1 + \chi^2)}{(1 + \chi^2)^2} U = 0, \tag{15}$$

where $\chi = rk/m$. This equation differs from the usual Suydam equation (2) only in that $\gamma + \delta$ appears instead of γ. Thus, instead of the usual Suydam stability condition $\gamma < 1/4$, for a toroidal cord we obtain $\gamma + \delta < 1/4$. In expanded form the stability condition may be written as follows:

$$-\frac{8\pi p'}{r H_z^2} - \frac{8\pi p' U'}{H_\varphi^2 U} + \frac{1}{2} \left(\frac{8\pi p'}{H_\varphi^2} \right)^2 \left(\frac{r}{R_0} \right)^2 < \frac{1}{4} \left(\frac{q'}{q} \right)^2. \tag{16}$$

A qualitatively similar criterion was obtained in (8).

As shown by V. D. Shafranov, Tokamak-type systems possess a minimum of the mean magnetic field in the sense that $U' < 0$, with

$$-\frac{U'}{U} = \frac{4r}{R_0^2} \left(\Lambda(r) + \frac{5}{4} \right) + \frac{r^2}{R_0^2} \Lambda, \tag{17}$$

where Λ is the so-called coefficient of asymmetry of the azimuthal field (7).

It follows from relations (16), (17) that for $\beta_\varphi = 8\pi p / H_\varphi^2 < 1$ and a not too steep variation of p with r, the interchange instability is stabilized.

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References

1. L. A. Artsimovich, *Controlled Thermonuclear Reactions*, Moscow, 1961.

2. B. Suydam, 2nd U. N. Int. Conf. Peaceful Uses of Atomic Energy, Geneva, 1958, p. 354.

3. H. Furth, J. Killen, M. Rosenbluth, B. Coppi, 2nd Conf. on Plasma, Phys. Paper CN 21/106, Culham, 1965.

4. R. Kulsrud, 2nd Conf. on Plasma. Phys. Paper CN 21/113, Culham, 1965.

5. P. H. Rebut, J. Nucl. Energy, Part C, 4, 159 (1962).

6. B. B. Kadomtsev, *Plasma Physics*, 4, USSR Academy of Sciences Press, 1958, p. 16.

7. V. D. Shafranov, *Problems of Plasma Theory*, 2, 1963.

8. A. A. Ware, Phys. Fluids, 7, No. 12, 2066 (1964).

Note: Figure translations are in progress. See original paper for figures.