ON THE LOSS OF THERMAL STABILITY OF THE MOTION OF A VISCO-PLASTIC MATERIAL

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Abstract

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ON THE LOSS OF THERMAL STABILITY OF THE MOTION OF A VISCO-PLASTIC MATERIAL

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In paper ([1]) the question of the so-called hydrodynamic thermal explosion was considered. In the present work an investigation is carried out of the possibility of a similar phenomenon in a visco-plastic medium, for which the deviator of the stress tensor is related to the deviator of the strain-rate tensor, according to ([2]), by a dependence of the form

$$D_\sigma = 2(\tau_s/H + \mu)D_\xi, \tag{1}$$

where H is the second invariant of the strain-rate tensor; τ_s is the limiting shear stress at which the medium begins to flow; μ is the dynamic coefficient of viscosity.

For the case of a steady, axisymmetric, laminar flow of an incompressible visco-plastic medium in an infinitely long circular pipe of radius r_0, with constant pressure gradient and density of the medium, the system of equations of motion and heat conduction, taking into account energy dissipation, will have the form

$$\frac{\partial}{\partial r}\left(-\tau_s + \mu\frac{\partial v}{\partial r}\right) - \frac{\tau_s}{r} + \frac{\mu}{r}\frac{\partial v}{\partial r} - \frac{\partial P}{\partial z} = 0,$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} - \frac{\tau_s}{\lambda I}\frac{\partial v}{\partial r} + \frac{\mu}{\lambda I}\left(\frac{\partial v}{\partial r}\right)^2 = 0. \tag{2}$$

Boundary conditions:

$$\text{for } r = r_0 \qquad v = 0, \qquad T = T_0;$$

$$\text{for } r = r_1 \qquad \partial v/\partial r = 0, \qquad \partial T/\partial r = 0, \tag{3}$$

or another condition may be imposed at the wall,

$$\text{for } r = r_0 \qquad \partial T/\partial r = hT, \tag{4}$$

where v is the velocity; $-\partial P/\partial z \equiv b$ is the pressure gradient; T is the temperature; T_0 is the temperature of the pipe walls; λ is the coefficient of thermal conductivity of the medium; I is the mechanical equivalent of heat; r_1 is the radius of the flow core, whose value is determined, according to ([2]), from the condition at the core boundary

$$b = 2\tau_s/r_1. \tag{5}$$

It is assumed that the viscosity and the yield limit are inversely proportional to the temperature,

$$\mu = \mu_0 a T^{-1}, \qquad \tau_s = \tau_0 a T^{-1}, \tag{6}$$

where a is a certain constant characterizing the medium.

Reducing the system of equations (2), with (6) taken into account, to dimensionless form, we have:

$$\frac{1}{x}\frac{d}{dx}\left[x\theta^{-1}\left(-\varepsilon + \frac{dw}{dx}\right)\right] + 1 = 0,$$

$$\frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - 4\varepsilon\delta\theta^{-1}\frac{dw}{dx} + 4\delta\theta^{-1}\left(\frac{dw}{dx}\right)^2 = 0, \tag{7}$$

where $x = r/r_0$, $w = \mu_0 v/(r_0^2 b)$, $\theta = T/a$ are dimensionless variables; $\delta = b^2 r_0^4/(4\lambda\Gamma\mu_0 a)$, $\varepsilon = \tau_0/(br_0)$ are dimensionless parameters.

The boundary conditions (3) are written in the form:

$$\text{at } x = 1 \qquad w = 0, \qquad \theta = T_0/a;$$

$$\text{at } x = x_1 \qquad dw/dx = 0, \qquad d\theta/dx = 0, \tag{8}$$

and condition (4):

$$\text{at } x = 1 \qquad d\theta/dx = r_0 h\theta. \tag{9}$$

Integrating once the first equation of system (7) and isolating the velocity gradient, we obtain

$$dw/dx = -\frac{1}{2}x\theta + \varepsilon, \tag{10}$$

where it has already been taken into account that the constant of integration is equal to zero as a consequence of the second boundary condition (8) and the condition at the core boundary (5), which in dimensionless form is written as

$$x_1 = 2\varepsilon/\theta, \tag{11}$$

where θ is evaluated at $x = x_1$.

Substitution of (8) into the second equation of system (7) leads to an inhomogeneous Bessel equation for determining the temperature θ outside the core

$$\frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} + \delta x^2\theta = 2\varepsilon\delta x \tag{12}$$

with boundary conditions (8) or with the wall condition (9).

As is known ([3,4]), the solution of equation (12) will be expressed in terms of Bessel functions of the first and second kind, and also in terms of Lommel functions:

$$\theta = c_1 J_0\left(\frac{\sqrt{\delta}}{2}x^2\right) + c_2 Y_0\left(\frac{\sqrt{\delta}}{2}x^2\right) + \varepsilon\delta^{1/4}\Gamma^2\left(\frac{3}{4}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\sqrt{\delta}}{4}x^2\right)^{3/2+2k}}{\Gamma^2\left({}^3/4 + k + 1\right)}. \tag{13}$$

The arbitrary constants c_1 and c_2 are determined from the boundary conditions (8) and from the condition that the determinant not be equal to zero,

$$\Delta = \sqrt{\delta}x_1\left[J_0(\alpha)Y_1(\alpha x_1^2) - Y_0(\alpha)J_1(\alpha x_1^2)\right], \tag{14}$$

where $\alpha = \sqrt{\delta}/2$.

Substituting expression (13) for the temperature into (10) and integrating, we find the velocity field

$$w = -\frac{c_1}{\sqrt{\delta}}\sum_{k=0}^{\infty}\frac{(-1)^k(\sqrt{\delta}\,x^2/4)^{2k+1}}{(k!)^2(2k+1)}$$

$$-\frac{2c_2}{\sqrt{\delta}}\sum_{k=0}^{\infty}\frac{(-1)^k(\sqrt{\delta}x^2/4)^{2k+1}}{(k!)^2(2k+1)}\left[\ln\left(\sqrt{\delta}\,\frac{x^2}{4}\right) + c - \frac{1}{2k+1} - \sum_{l=1}^{k}\frac{1}{l}\right]$$

$$-\delta^{-1/4}\varepsilon\Gamma^2\left(\frac{3}{4}\right)\sum_{k=0}^{\infty}\frac{(-1)^k(\sqrt{\delta}\,x^2/4)^{5/2+2k}}{(5/2+2k)\Gamma^2(^3/4+k+1)}+\varepsilon x+c_3, \qquad (15)$$

where c is Euler' s constant. The constant c_3 is determined from the first boundary condition (8).

In the case where the determinant Δ is equal to zero, the solution loses stability. The heat released due to viscous friction does not have time to be removed; rapid heating of the medium occurs, or the so-called hydrodynamic thermal explosion, although in the present case it would be convenient to call this phenomenon melting of the medium due to viscous friction outside the core.

Obviously, only the first root of the equation defining the critical state has meaning:

$$\Delta(\alpha)=0. \qquad (16)$$

In the case of boundary condition (9), to find the critical value of the parameter a we have the equation:

$$[2\alpha J_1(\alpha)-r_0hJ_0(\alpha)]Y_0(\alpha x_1^2)-[2\alpha Y_1(\alpha)-r_0hY_0(\alpha)]J_0(\alpha x_1^2)=0. \qquad (17)$$

Specifying the pressure gradient b completely determines the value x_1 from the condition on the core boundary (5) or (11) and the temperature solution (13). In the present case it is more convenient to prescribe the value of x_1 in order to compute the critical value of the parameter δ.

Let $x_1=1/2$ or $x_1=1/4$. Then, based on the tables ([3]), we find that the roots of equation (16) will be, respectively, ~ 2.7 and ~ 2.5, and hence $\delta=29.2$ and 2.5. As the core decreases, i.e., as the pressure gradient increases, the critical value of δ decreases.

It is interesting to note that in the case of constant values of μ and τ, critical conditions are impossible. Indeed, in this case the third term on the left in equation (12) will not contain θ, and the equation is integrated ([4]) in elementary functions:

$$T=-\delta x^4+\frac{32}{9}\varepsilon\delta x^3-\left(\frac{32}{3}\varepsilon\delta x_1^3-4\delta x_1^4\right)\ln x+\delta-\frac{32}{9}\varepsilon\delta+T_0. \qquad (18)$$

It is evident from the solution that the parameter δ enters it linearly and has no critical values.

It is clear that the stronger the dependence of μ and τ on temperature—for example, a power law with $n<-1$, or an exponential dependence—the more rapidly the critical flow conditions will arise.

The theory described may be applied in lubrication theory and in the theory of extrusion of a viscoplastic material through orifices with very large pressure gradients.

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Note: Figure translations are in progress. See original paper for figures.