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Abstract

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MATHEMATICS

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ELLIPTIC OPERATORS AND SUBMANIFOLDS

The present work is a direct continuation of the authors' work (2). We shall not here redefine the operators (A^{-1}, SD_F), the semigroups $\mathrm{Op}^{s,t}(X,\xi)$, $\mathrm{Ell}^{s,t}(X,\xi)$ for a vector bundle ξ of dimension $2q$, the "trace" operator i_{SD}, the character of an operator $\mathrm{Ch}\,A \in H^*(X,Q)[t]/t^{|s,t-q+1}$, the Pontryagin character $\mathrm{Ph}(n) \in H^*(Y,Q)[t]$, and the restriction

$$i^* : H^*(X,Q)[t]/t^{l-q+1} \to H^*(Y,Q)[t]/t^{l-q-q'+1}.$$

We introduce here an important special case of the operators SD_F: let X be a Kähler manifold, $Y \subset X$ a complex submanifold, and n the complex normal bundle. Let

$$F = \sum_j S^j n \subset \sum_j S^j cn;$$

then SD_F shall be denoted by SD_c.

By analogy with (2), for a (Kähler) bundle ξ over X of dimension q, a semigroup of operators $\mathrm{Op}_c^{s,t}(X,\xi)$ is introduced:

$$A = (A^{jk}) : \sum_{j \le l\, s,t-q} \Gamma^{s-q-j}(X, S^j\xi \otimes E_1) \to \sum_{k \le l\, s,t-q} \Gamma^{t+q+k}(X, S^j\xi \otimes E_2)$$

and the homomorphic trace operator is defined

$$i_{SD_c} : \mathrm{Op}_c^{s,t}(X,\xi) \to \mathrm{Op}_c^{s,t}(Y, n \oplus i^*\xi),$$

by setting

$$i_{SD_c}A = (i_{SD_c}A^{jk}) \quad \text{and} \quad i_{SD_c}A^{jk} = i_c^* \circ A^{jk} \circ \chi_c.$$

Here it is necessary to explain more precisely what the operators i^* and χ in general are. The restriction with normal derivatives i^* is the operator which assigns to each smooth section of the bundle $S^j cn$ on Y a complex-valued function on Y, depending linearly on this section, according to the following rule: we differentiate the section on X along some extension to a neighborhood $U \supset Y$ of the original section of the bundle $S^j cn$, and then restrict to Y the result of differentiation. We obtain a linear form

$$\Gamma(Y, S^j cn) \to \Gamma(Y, 1),$$

which may be regarded as a section of the bundle $S^j cn$ on Y. The operator χ is adjoint to i^*. The operators i_c^* and χ_c are obtained analogously, but one must restrict oneself only to complex differentiations.

We now turn to our main result in K-theory. Recall some standard notation: $M\eta$ is the Thom complex; $j : B \to M\eta$ is the inclusion of the base B; $i_\tau : Mi^*\eta \to M\eta$ for $i : Y \to X$ is the natural map of Thom complexes. If $\eta = \eta_X$ is the cotangent bundle, then $i^*\eta_X = \bar{n} \oplus \eta_Y$, and we have:

$$(M\eta_Y, *) \xrightarrow{j} (Mi^*\eta_X, *) \xrightarrow{i_\tau} (M\eta_X, *),$$

where $(Mi^*\eta_X, *)$ is the relative Thom complex of the bundle $\bar{n}$ over the pair

$$(M\eta_Y, *) = (B(Y), S(Y)).$$

We define mappings:

$$S_t : K(X) \to K(X)[t], \qquad \lambda_{-t} : K(X) \to K(X)[t],$$

$$S_t^R = S_t \circ c : KO(X) \to K(X)[t], \qquad \lambda_{-t}^R = \lambda_{-t} \circ c : KO(X) \to K(X)[t],$$

where

$$S_t\xi = \sum_{j \geq 0} S^j \xi \, t^j; \qquad \lambda_{-t}\xi = \sum_{j \geq 0} (-1)^j \Lambda^j \xi \, t^j;$$

$$S_t\xi \circ \lambda_{-t}\xi = 1; \quad S_t\xi \oplus \eta = S_t\xi \circ S_t\eta; \quad \lambda_{-t}\xi \oplus \eta = \lambda_{-t}\xi \circ \lambda_{-t}\eta;$$

$$\delta : \mathrm{Ell}^{s,t}(X, \xi) \to K(M\eta_X, *)[t]/t^{s,t-q+1};$$

$$\delta_c = \delta \circ r : \mathrm{Ell}_c^{s,t}(X, \xi) \to K(M\eta_X, *)[t]/t^{s,t-q+1};$$

$$2q = \dim_R \xi, \qquad \delta(A) = \sum_j \alpha(\tilde{\delta}_j) t^j;$$

$\tilde{\delta}_j$ are the elliptic operators indicated in (2) for A;
$\alpha(\tilde{\delta}_j) \in K(M\eta_X, *)$ are their Atiyah-Singer elements. Then

$$\varphi^{-1} \operatorname{ch} \delta(A) = \sum_j \varphi^{-1} \operatorname{ch} \alpha(\tilde{\delta}_j) t^j = \operatorname{Ch} A, \quad \operatorname{ch} S_t^R \xi = \operatorname{Ph} \xi, \quad \operatorname{ch} S_t \xi = \operatorname{Ch} \xi.$$

Put

$$i_M^* = j^* \circ i_T^*; \qquad K(M\eta_X, *)[t]/t^{l-q+1} \to K(M\eta_Y, *)[t]/t^{l-q-q'+1}$$

naturally.

The following contravariant theorem (of Riemann-Roch type) holds:

Theorem 1. a) If

$$A \subset \operatorname{Ell}^{s,t}(X, \xi) \cap i_{SD}^{-1} \operatorname{Ell}^{s,t}(Y, \eta \oplus i^* \xi),$$

then the relation

$$\lambda_{-t}^R \eta \oplus i^* \xi \circ \delta(i_{SD} A) = i_M^* (\lambda_{-t}^R \xi \circ \delta(A))$$

is satisfied;

b) if

$$A \in \operatorname{Ell}_c^{s,t}(X, \xi) \cap i_{SD_c}^{-1} \operatorname{Ell}_c^{s,t}(Y, \eta \oplus i^* \xi),$$

then the relation

$$(\lambda_{-t}) \eta \oplus i^* \xi \circ \delta_c(i_{SD_c} A) = i_M^* (\lambda_{-t} \xi \circ \delta_c(A)),$$

is satisfied, where

$$i_M^* = j^* \circ i_T^*, \quad i_T^*, \quad j^*$$

were defined above.

Thus, for $\operatorname{Ell}^{s,t}(X, \xi)$, the functorial expression is $\lambda_{-t}^R \xi \circ \delta(A)$ for the case of immersions, moreover contravariantly.

Corollary 1. a)

$$\operatorname{Ch}(i_{SD} A) = \chi(\eta)\, (i^* \operatorname{Ch} A) \operatorname{Ph}(\eta);$$

b)

$$\operatorname{Ch}(i_{SD_c} A) = c_q(\eta) \circ i^* (\operatorname{Ch} A) \operatorname{Ch}(\eta).$$

Denote the expression $\lambda_{-t}^R \xi \circ \delta(A)$ by

$$R(A) \in K(M\eta_X, *)[t]/t^{l-q+1}.$$

The index formula can be written as follows:

$$I_a(A) = \left\{ [\varphi^{-1} \operatorname{ch}(S_t^R \xi R(A))] \circ T(c\eta_X), [Y] \right\}_{t=1}.$$

Since

$$I_a(A^{-1}, SD) - I_a(A^{-1}) = I_a(i_{SD}A)$$

(see (2)), we have

$$I_a(A^{-1}, SD) - I_a(A^{-1}) =$$
$$= \{[\varphi^{-1}\operatorname{ch}(i_M^* R(A)) \circ S_t^R \eta \oplus i^*\xi] \circ T(c\eta_Y), [Y]\}_{t=1}.$$

and analogously for SD_c.

Let us now pass to the analytic meaning of the results and to general boundary operators associated with Y. Let

$$C : \Gamma^s(X, E) \to \Sigma\Gamma^{s_j}(Y, E_j)$$

and

$$B : \Sigma\Gamma^{t_j}(Y, E_j) \to \Gamma^t(Y, E')$$

be operators with the following properties:

1. The image of B consists only of sections concentrated on Y; all sections that vanish on any neighborhood $U \supset Y$ belong to the kernel of C.
2. The operators B and C have "symbols" :

$$\sigma_B(\tau, \xi) : \sum_j \pi^*_{i^*\eta_X} E'_j \to \pi^*_{i^*\eta_X} E', \qquad \sigma_C(\tau, \xi) : \pi^*_{i^*\eta_X} E \to \sum_j \pi^*_{i^*\eta_X} E_j.$$

3. B is a monomorphism and C is an epimorphism; here τ are coordinates in the fibration $\bar{\eta}$ on Y, and ξ are coordinates in the fibration η_Y; $\pi_\eta : B(\eta) \to B$ (for a fibration η with base B) is the projection.

Let

$$A : \Gamma^t(X, E') \to \Gamma^s(X, E)$$

be an operator and let σ_A be its symbol,

$$\sigma_A : \pi^*_{\eta_X} E' \to \pi^*_{\eta_X} E.$$

The symbols are not assumed to be isomorphisms of fibrations.

Lemma 1. The symbol of the operator $C \circ A \circ B$ is computed by the formula

$$\sigma_{C\circ A\circ B}(\xi) = \int_{\xi=\text{const}} \sigma_C \circ \sigma_A \circ \sigma_B \, d\tau,$$

where the integration is over the linear mapping depending on the parameter ξ.

An important example: $C = i^*$, $B = \varkappa$; here we have

$$\sigma_C(\tau, \xi)[e] = e \otimes \sum_{|m| \le l-q} \tau^m,$$

where $m = (m_1, \dots, m_k)$, $\tau^m = \tau_1^{m_1} \circ \dots \circ \tau_k^{m_k}$, $|m| = \sum_j m_j$; e is a vector of the quotient bundle $\pi^*_{i^*\eta X}E$;

$$\sigma_B(\tau, \xi)[P \otimes e'] = P(\tau)e',$$

where e' is a vector of the quotient bundle $\pi^*_{i^*_\eta X}E'$ and P is a polynomial of degree $\leq l_s, t - q$ on the quotient bundle $\bar{n}$. Similarly for $C = i^*_c,\ B = \varkappa_c$.

Definition 1. A direction (τ) is called **elliptic with respect to the triple of operators** (A, B, C) if, for any $\xi \neq 0$, the mapping

$$\int_{\substack{\xi \neq 0 \\ \xi = \text{const}}} \sigma_C \circ \sigma_A \circ \sigma_B \, d\tau$$

is an isomorphism at this point.

We obtain important special cases when $B = \varkappa,\ C = i^*$ or $B = \varkappa_c,\ C = i^*_c$. In these cases we shall call directions elliptic with respect to the triple (A, B, C), A-**elliptic**, respectively, in the real and complex sense.

Definition 2. A submanifold $Y \subset X$ is called **elliptic with respect to the triple** (A, B, C) if

$$\operatorname{codim}_R Y = \dim_R \tau$$

and at each point $y \in Y$ this submanifold is transversal to the elliptic direction τ.

Let the operator A be elliptic, and let A^{-1} be its inverse (mod Comp). Consider the boundary operator

$$(A^{-1}, B, C) : \Gamma^s(X, E) \to \Gamma^t(X, E') / \operatorname{Im} B \oplus \sum_j \Gamma^{s_j}(Y, E_j),$$

putting

$$(A^{-1}, B, C)[u] = (A^{-1}u \, [\mathrm{mod} B] \oplus Cu).$$

It should be noted that operators of the more general type

$$\begin{pmatrix} A^{-1} & B \\ C & D \end{pmatrix} : \Gamma^s(X, E_1) \oplus \Sigma\Gamma^{t_k}(Y, F'_k) \to \Gamma^t(X, E_2) \oplus \Sigma\Gamma^{s_j}(Y, F_j),$$

where

$$\begin{pmatrix} A^{-1} & B \\ C & D \end{pmatrix} [u \oplus v] = (Au + Bv \oplus Cu + Dv)$$

and A^{-1} is elliptic, also fall under our scheme. Put

$$\sigma = \sigma_D - \int \sigma_C \circ \sigma_A \circ \sigma_B \, d\tau : \pi^*_{\eta Y}(\Sigma F_k) \to \pi^*_{\eta Y}(\Sigma F'_j).$$

The following simple result holds.

Theorem 2.

$$I_a \begin{pmatrix} A^{-1} & B \\ C & D \end{pmatrix} = I_a(A^{-1}) + I_a(D - C \circ A \circ B),$$

where $\sigma_{D - C\circ A\circ B} = \sigma$; the case of the triple (A^{-1}, B, C) corresponds to $D = 0$, and the case of operators of type (A^{-1}, SD_F) reduces to operators of the form

$$\begin{pmatrix} A^{-1} & \varkappa_F \\ i^*_F & 0 \end{pmatrix}.$$

We now return to the operators $A \subset \mathrm{Op}^{s,t}(X, E)$ or $A \in \mathrm{Op}^{s,t}_c(X, E)$. Let τ be a $2q'$-dimensional direction in the real sense at a point $x \in X$, and let E_x be the quotient bundle E; E_{1x} and E_{2x} are the quotient bundles E_1, E_2 at the points x_1, where E_1 and E_2 are defined by the operator A. We state an assertion useful from the point of view of geometric interpretation.

Lemma 2. The integral

$$\int_{\xi = \mathrm{const}} \sigma_{i^*} \circ \sigma_A \circ \sigma_\varkappa \, d\tau$$

defines a bilinear product

$$[a, b] \in \mathrm{Hom}_c(E_1, E_2)_x,$$

where

$$a, b \in \sum_{j \le l - q - q'} S^j c(\tau \oplus E_x);$$

it is symmetric for $E = 0$, Hermitian for $i^*_c, \varkappa_c$ and Kähler quotient bundles E_1, E_2, E, if the operator A is naturally related to the Kähler structure. If this product

$$[a, b] \in \mathrm{Hom}_c(E_1, E_2)_x$$

is nondegenerate in the sense that, for any

$$a \in \sum_j S^j c(\tau \oplus E_x),$$

we have

$$\bigcap_{b \in S^1 c(\tau \oplus E_x)} \mathrm{Ker}[a, b]^j = 0,$$

then the direction (τ) is elliptic relative to the operator A (the triple A, i^*, χ) or (A, i_c^*, χ_c).

Types of operators.

1. Real operators $\mathrm{Op}_R^{s,t}(X, E) \xrightarrow{c} \mathrm{Op}^{s,t}(X, E)$ correspond to the fact that $E_1 = cE_1'$, $E_2 = cE_2'$ and the operator A is the complexification of a real one. Then the bilinear product $[a, b]$ indicated in Lemma 2 is real if $a, b \in \Sigma S^j(\tau \oplus E_x)$. For scalar operators $E_1' = E_2' = R$ the product $[a, b]$ is scalar and positive definite.

2. Hermitian operators $A \in \mathrm{Op}_c^{s,t}(X, \xi)$, for which the product $[a, b]$ is Hermitian, $a, b \in \Sigma S^j(\tau \oplus E)$, although it may also be expressed. The collection of such operators gives the Dolbeault operator (see below) on forms with coefficients in bundles. Here there is also the scalar case $E = 0$, $E_1 = E_2 = C$ (more generally, E_1 and E_2 are one-dimensional). In all scalar cases the topological invariants of the operator $i_{SDc}A$, $i_{SD}A$ are equal to zero if $\dim Y < \dim X$.

3. Group operators with coefficients in a bundle (operators associated with a G-structure in the terminology of ([1]). Let us indicate examples:

a) The Hirzebruch operator

$$E_1 = \Lambda^+ \mathrm{cn}_X \otimes E, \qquad E_2 = \Lambda^- \mathrm{cn}_X \otimes E, \qquad A^{-1} = (d + \delta)\Delta^m \otimes 1,$$

$$A : \Gamma^{-m-1/2}(X, E_1) \to \Gamma^{m+1/2}(X, E_2), \qquad A \in \mathrm{Op}^{-m-1/2, m+1/2}(X, 0).$$

Denote the element $a(A) \in K(M\eta_X, *)$ by $h_m(X, E)$, and for the adjoint operator A^* the element $a(A^*)$ by

$$h_m^*(X, E) = -h_m(X, E), \qquad h_m(X, E) = \delta(A)\big|_{t=1}.$$

b) The Euler operator:

$$E_1 = \Lambda^{\mathrm{even}} \mathrm{cn}_X \otimes E, \qquad E_2 = \Lambda^{\mathrm{odd}} \mathrm{cn}_X \otimes E, \qquad A^{-1} = (d + \delta)\Delta^m,$$

$$A : \Gamma^{-m-1/2}(X, E_1) \to \Gamma^{m+1/2}(X, E_2), \qquad a(A) = \chi_m(X, E),$$
$$a(A^*) = \chi_m^*(X, E) = -\chi_m(X, E).$$

The Euler operator is real if $E = cE'$. Here $\chi_m(X, E) = \delta(A)\big|_{t=1}$.

c) The Todd operator

$$E_1 = \Lambda^{\text{even}}\eta_X \otimes E, \qquad E_2 = \Lambda^{\text{odd}}\eta_X \otimes E; \qquad A^{-1} = (\partial + \bar{\partial})\Box^m \otimes 1;$$

$$A : \Gamma^{-m-1/2}(X, E_1) \to \Gamma^{m+1/2}(X, E_2); \qquad A \in \mathrm{Op}^{-m-1/2, m+1/2}(X, 0);$$
$$a(A) = t_m(X, E); \qquad a(A^*) = t_m^*(X, E) = -t_m(X, E);$$

X, E, E_1, E_2 are Kähler; $t_m(X, E) = \delta(A)\big|_{t=1}$.

d) The Dirac operator:

$$E_1 = \Delta^+\eta_X \otimes E, \qquad E_2 = \Delta^-\eta_X \otimes E; \qquad A^{-1} = (d + d^*)[d + d^*]^{2m} \otimes 1,$$

where

$$d : \Gamma(X, \Delta^+\eta_X) \to \Gamma(X, \Delta^-\eta_X)$$

is the ordinary Dirac operator;

$$A : \Gamma^{-m-1/2}(X, E_1) \to \Gamma^{m+1/2}(X, E_2);$$

X, E, E_1, E_2 are spinor. Put

$$a(A) = d_m(X, E), \qquad a(A^*) = d_m^*(X, E) = -d_m(X, E).$$

From the preceding results one obtains, respectively, the following formulas for the embedding $i : Y \subset X$, $\operatorname{codim}_R Y = 2q$:

a) $$\lambda_-^R tn{\circ}\delta(i_{SD}A) = \sum_{j \le m-q} t^j \left\{h_{m-q-j}\left(Y, \overline{\Lambda cn} \otimes i^*E\right) + h^*_{m-q-j}\left(Y, \Lambda^+ cn \otimes iE\right)\right\},$$

b) $$\lambda_-^R tn \circ \delta(i_{SD}A) = 0 \quad \text{for the Euler operator, } \dim Y < \dim X,$$

c) $$\lambda\_tn{\circ}\delta(i_{SD_c}A) = \sum_{j \le m-q} t^j \left\{t_{m-q-j}\left(Y, \Lambda^{\text{odd}}\eta \otimes i^*E\right) + t^*_{m-q-j}\left(Y, \Lambda_n^{\text{even}} \otimes i^*E\right)\right\},$$

d) $$\lambda_-^R tn{\circ}\delta(i_{SD}A) = \sum_{j \le m-q} \left\{d_{m-q-j}\left(Y, \Delta^- n \otimes i^*E\right) d^*_{m-q-j}\left(Y, \Delta^+ n \otimes i^*E\right)\right\} t^j,$$

where E, Y, X are spinor. Note that $\chi_m(X, E)$, $h_m(X, E)$, $t_m(X, E)$, $d_m(X, E)$ do not depend on m.

It is not difficult to expand these formulas for the cases $2 \dim Y = \dim X$ and $\operatorname{codim}_R Y = 2$. For example, in the first case one must use Corollary 1 of the authors' paper $(^2)$ and the fact that

$$\mathrm{ch}^0 A = 2^n,\ 0,\ (-1)^n,\ 1$$

respectively in cases a), b), c), d). For lack of space we do not present them.

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1. M. F. Atiyah, I. M. Singer, *Bull. Am. Math. Soc.*, **69**, No. 3, 422 (1963).
2. S. P. Novikov, B. Yu. Sternin, DAN, **170**, No. 6 (1966).
3. B. Yu. Sternin, *Tr. Mosk. matem. obshch.*, **15** (1966).

Note: Figure translations are in progress. See original paper for figures.