ON LOCALLY NILPOTENT SUBGROUPS OF THE INFINITE SYMMETRIC GROUP

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ON LOCALLY NILPOTENT SUBGROUPS OF THE INFINITE SYMMETRIC GROUP

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Let X be an arbitrary nonempty set; $S(X)$ the group of all bijective mappings $f : X \to X$; $SF(X)$ the subgroup of $S(X)$ consisting of all such $f \in S(X)$ that $f(x) \neq x$ only for finitely many points $x \in X$.

In the present paper it is proved that, in the case of infinite X, every transitive locally nilpotent subgroup of the group $SF(X)$ is a p-group. Nilpotent subgroups of the group $SF(X)$ are described. One property of stationary subgroups of transitive nilpotent permutation groups of finite degree is given.

I. If $\mathfrak{G}$ is an arbitrary group, then every homomorphism $u : \mathfrak{G} \to S(X)$ will be called a **representation** of the group $\mathfrak{G}$. Two representations $u : \mathfrak{G} \to S(X)$ and $v : \mathfrak{G} \to S(Y)$ are called **equivalent** if there exists a bijective mapping $\varphi : X \to Y$ such that, for every a in $\mathfrak{G}$, $v(a) = \varphi u(a)\varphi^{-1}$. Let $\mathfrak{H}$ be an intransitive subgroup of $S(X)$, and let X_α be one of the systems of imprimitivity of the group $\mathfrak{H}$. We introduce the transitive representation r_α of the group $\mathfrak{H}$ by putting

$$r_\alpha : \mathfrak{H} \to S(X_\alpha), \qquad r_\alpha(h) = h|X_\alpha, \tag{1}$$

where $h|X_\alpha$ is the restriction of h to X_α, $h \in \mathfrak{H}$.

Let $\mathfrak{G}$ be a transitive subgroup of the group $S(X)$; let $\mathfrak{H}$ be a subgroup of its centralizer $C(\mathfrak{G})$ in $S(X)$. Then the following two lemmas are valid.

Lemma 1. If the group $\mathfrak{H}$ is transitive, then on the set X one can define a group operation $+$ such that the group $\mathfrak{G}$ turns out to be the left regular representation of the group $\langle X, +\rangle$, and $\mathfrak{H}$ the right regular representation of the group $\langle X, +\rangle$.

Lemma 2. If $\mathfrak{H}$ is intransitive, and $X = \bigcup_{\alpha\in I} X_\alpha$ is the decomposition of X into systems of imprimitivity of $\mathfrak{H}$, then for any $\alpha, \beta \in I$ the transitive representations r_α, r_β of the group $\mathfrak{H}$, defined by formulas (1), are equivalent.

From Lemma 2 it follows:

Lemma 3. If X is an infinite set, and Γ is a transitive subgroup of the group $SF(X)$, then the center of the group Γ coincides with the identity group.

II. **Lemma 4.** Let X be an arbitrary nonempty set, and let $\mathfrak{P}$ be a transitive p-subgroup of the group $SF(X)$. If a locally nilpotent subgroup Γ of the group $SF(X)$ contains $\mathfrak{P}$, then Γ is also a p-group. In particular, a transitive Sylow p-subgroup of the group $SF(X)$ is a maximal locally nilpotent subgroup of the group $SF(X)$.

Proof. For finite X the lemma was proved in the paper ([1]). Let X be an infinite set, and let Γ be a locally nilpotent subgroup of $SF(X)$ containing $\mathfrak{P}$. Suppose further, contrary to the assertion of the lemma, that Γ is not a p-group. Then, by virtue of the periodicity of Γ, in Γ there is an element a of prime order $q \neq p$. Since Γ is locally nilpotent, the cyclic group $A = (a)$ is contained in the centralizer of the group $\mathfrak{P}$ in

$S(X)$. If now $X = \bigcup_{\alpha\in I} X_\alpha$ is the decomposition of X into the systems of imprimitivity of the group A, then, according to Lemma 2, the representations $r_\alpha : A \to S(X_\alpha)$, where $r_\alpha(a) = a/X_\alpha$, are pairwise equivalent. Only two cases are possible: either each X_α consists of q points, permuted cyclically by a/X_α, or each X_α consists of one point. In the first case a leaves fixed no point of the set X, which contradicts the inclusion $a \in SF(X)$; in the second, a leaves every point of the set X fixed. The latter contradicts the fact that the order of a is equal to the prime number q. The lemma follows.

Theorem 1. *Let X be an infinite set*. Then every transitive locally nilpotent subgroup of the group $SF(X)$ is a p-group. In particular, a maximal transitive locally nilpotent subgroup of $SF(X)$ is a Sylow p-subgroup of the group $SF(X)$.*

Proof. Let Γ be a transitive locally nilpotent subgroup of $SF(X)$. If one of the Sylow p-subgroups of the group Γ is transitive, then the theorem follows from Lemma 4. We first show that the number of systems of imprimitivity of a Sylow p-subgroup of the group Γ is finite. Let $\mathcal{H}$ be a nonidentity Sylow p-subgroup of the group Γ, and let $X = \bigcup_{\alpha\in I} X_\alpha$ be the decomposition of X into the systems of imprimitivity of the group $\mathcal{H}$. Consider the transitive representations r_α of the group $\mathcal{H}$

$$r_\alpha : \mathcal{H} \to S(X_\alpha), \qquad r_\alpha(h) = h_\alpha = h/X_\alpha, \quad h \in \mathcal{H}. \tag{2}$$

For any α and β in I, the representations r_α and r_β are equivalent. Indeed, Γ can be represented as the direct product

$$\Gamma = \mathcal{H}\mathcal{D}, \tag{3}$$

where $\mathcal{D}$ is a normal divisor of Γ, the orders of whose elements are relatively prime to p. Since Γ is transitive, by virtue of (3), for any $\alpha, \beta \in I$ we have $X_\beta = d(X_\alpha)$, where $d \in \mathcal{D}$. Then for the restriction $h_\beta = h/X_\beta$, where $h \in \mathcal{H}$, one can write $h_\beta(x_\beta) = h(x_\beta) = hd(x_\alpha) = dh(x_\alpha) = dhd^{-1}(x_\beta) = d_\alpha h_\alpha d_\alpha^{-1}(x_\beta)$, where $x_\alpha \in X_\alpha$, $x_\beta \in X_\beta$, $d_\alpha = d/X_\alpha$. Hence $h_\beta = d_\alpha h_\alpha d_\alpha^{-1}$, $r_\beta(h) = d_\alpha r_\alpha(h) d_\alpha^{-1}$. The equivalence of r_α and r_β is proved. From the pairwise equivalence of the representations r_α and the inclusion $\mathcal{H} \subset SF(X)$, it follows that the set of systems of imprimitivity of the group $\mathcal{H}$ is finite. Hence it follows in turn that every nonidentity Sylow p-subgroup of the group Γ is infinite.

We now show that a nonidentity Sylow p-subgroup $\mathcal{P}$ of the group Γ is transitive. Suppose $\mathcal{P}$ is intransitive. Then X is decomposed into a finite number of systems of imprimitivity of the group $\mathcal{P}$:

$$X_1 \cup \ldots \cup X_k = X, \qquad k > 1. \tag{4}$$

Since $\mathcal{P}$ is a normal divisor of Γ, (4) is a decomposition of X into systems of imprimitivity of Γ.

Consider now the homomorphism $\gamma : \Gamma \to S_k$, where S_k is the symmetric group of degree k, permuting the systems $X_1, \ldots, X_k$ among themselves.

Let N be the kernel of the homomorphism γ. We show that $N = \mathcal{P}$. Obviously, Γ is represented as the direct product $\Gamma = \mathcal{P}U$, where the orders of the elements of the direct factor U are relatively prime to p. It is clear that N contains $\mathcal{P}$. Consequently, it is enough to show that N has no nonidentity element u from U. Let $u \in N \cap U$. Then for $j = 1, \ldots, k$, $(u_j)\mathcal{P}_j$, where $u_j = u/X_j$, $\mathcal{P}_j = \mathcal{P}/X_j$, is a locally nilpotent transitive subgroup of the group $SF(X_j)$, containing the transitive p-group $\mathcal{P}_j$. According to Lemma 4, $(u_j)\mathcal{P}_j$ is also a p-group. Consequently, u is a p-element. Since U has no nonidentity p-elements, $u = 1$, $N = \mathcal{P}$. From the equality $N = \mathcal{P}$ it follows that U is a finite subgroup of Γ. Consequently, every Sylow q-subgroup of the group Γ contained in U is finite. But above

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* As follows from paper (1), in the case of finite X Theorem 1 is false.

we have proved that in Γ there are no nonidentity finite Sylow q-subgroups. Thus the assumption that $\mathcal{G}$ is intransitive leads to a contradiction. Hence $\mathcal{G}$ is transitive. The theorem follows from this.

Corollary. Let $\mathcal{G}$ be a maximal locally nilpotent subgroup of the group $SF(X)$, where X is an infinite set. Then $\mathcal{G}$ is the direct product of its restrictions $\mathcal{G}_\alpha$ to the transitivity systems X_α of the group $\mathcal{G}$. For infinite X_α the group $\mathcal{G}_\alpha$ is a Sylow p-subgroup of the group $SF(X_\alpha)$, and for finite X_α, $\mathcal{G}_\alpha$ is a maximal transitive nilpotent subgroup of $S(X_\alpha)$, described in (1).

The Sylow p-subgroups of $SF(X)$ for countable X were studied by I. D. Ivanyuta in the paper (3).

III. From Lemma 3 and the results of paper (2) it follows

Theorem 2. Let X be an infinite set, and let $\mathcal{G}$ be a nilpotent subgroup of the group $SF(X)$. Then $\mathcal{G}$ is intransitive, and every one of its transitivity systems X_α is finite. $\mathcal{G}$ is maximal among the nilpotent subgroups of $SF(X)$ if and only if the following four conditions hold simultaneously:

1) $\mathcal{G}$ is the direct product of its restrictions $\mathcal{G}_\alpha = \mathcal{G}/X_\alpha$ to the transitivity systems X_α;
2) $\mathcal{G}_\alpha$ is a maximal transitive nilpotent subgroup of the group $S(X_\alpha)$;
3) the number of transitivity systems of one and the same order p^t, where p is a prime number and $t > 0$, is less than the number p;
4) among the transitivity systems of the group $\mathcal{G}$ there is at most one system consisting of a single point.

Analogously, with the aid of Lemma 3, one can obtain

Proposition. Every ZA-group contained in $SF(X)$ is nilpotent.

IV. We give one property of nilpotent transitive subgroups of the finite symmetric group S_n of degree n.

If n is a natural number, then denote by the letter m the product of all distinct prime divisors of the number n.

Theorem 3. Let Γ be a maximal transitive nilpotent subgroup of $S_n = S(X)$. If Γ_1 is the stabilizer subgroup of the group Γ; Y is the set of all such x in X that $g_1(x) = x$ for every substitution $g_1 \in \Gamma_1$, then the number of points of the set Y coincides with the number m.

Corollary. Let Γ be a transitive nilpotent subgroup of the group $S_n = S(X)$. Then a substitution g from Γ that leaves one point $x \in X$ fixed leaves fixed at least m points.

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Note: Figure translations are in progress. See original paper for figures.