GEOMETRIC THEORY OF UNITARY EQUIVALENCE OF MATRICES

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Abstract

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GEOMETRIC THEORY OF UNITARY EQUIVALENCE OF MATRICES

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1. Let $\mathfrak{A} = \{A_1, A_2, \dots, A_m\}$ and $\mathfrak{B} = \{B_1, B_2, \dots, B_m\}$ be two sequences of length m of matrices of order n over the field of complex numbers. $\mathfrak{A}$ and $\mathfrak{B}$ are called **unitarily equivalent** if there exists a unitary matrix S such that $B_i = S^{-1}A_{iS}$ for all $i = 1, 2, \dots, m$. For the case when $m = 1$ and the matrices A_1 and B_1 are normal, a necessary and sufficient condition for unitary equivalence (u.e.) is the coincidence of the eigenvalues. In recent years numerous papers have been devoted to the general case (^{1}—^{7}). In the present note this question is treated in terms of n-dimensional unitary geometry. We indicate a procedure for the successive computation of numerical invariants for $\mathfrak{A}$ and $\mathfrak{B}$, whose coincidence is necessary and sufficient for their u.e. Our results are analogous to those obtained in (^{7}), but the geometric interpretation, in our view, is more transparent and admits far-reaching generalizations.

2. One may restrict oneself to the case when all the matrices A_i and B_i are normal. Indeed, every matrix A admits a unique representation in the form $A = A^{(1)} + iA^{(2)}$, where $A^{(1)}$ and $A^{(2)}$ are Hermitian and, consequently, normal. For the u.e. of the sequences $\mathfrak{A}$ and $\mathfrak{B}$ it is necessary and sufficient that the sequences of length $2m$, $\{A_i^j\}$, $\{B_i^{(j)}\}$, $i = 1, 2, \dots, m;\ j = 1, 2$, be u.e. Thus, we assume from the outset that $\mathfrak{A}$ and $\mathfrak{B}$ consist of normal matrices.

3. To each matrix A_i (respectively B_i) there corresponds a set of eigenvalues and an orthogonal decomposition of the unitary space V of dimension n into a sum of admissible subspaces whose dimensions are equal to the multiplicities of the corresponding eigenvalues.

Thus, the sequence $\mathfrak{A}$ determines m orthogonal decompositions

$$\begin{aligned}
&V = U_{11} \perp U_{12} \perp \dots \perp U_{1r_1},\\
&V = U_{21} \perp U_{22} \perp \dots \perp U_{2r_2},\\
&\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&V = U_{m1} \perp U_{m2} \perp \dots \perp U_{mr_m}
\end{aligned} \tag{1}$$

and, analogously, for the sequence $\mathfrak{B}$,

$$\begin{aligned} V &= T_{11} \perp T_{12} \perp \dots \perp T_{1s_1}, \\ V &= T_{21} \perp T_{22} \perp \dots \perp T_{2s_2}, \\ &\dots\dots\dots \\ V &= T_{m1} \perp T_{m2} \perp \dots \perp T_{ms_m}. \end{aligned} \tag{2}$$

A series of orthogonal decompositions of the type (1) or (2) will be called a **configuration**; the orthogonal summands U_{ij} and T_{ij}, its **components**.

Theorem 1. *For the unitary equivalence of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *it is necessary and sufficient: a) that the eigenvalues of the matrices* A_i *and* B_i *coincide (including their multiplicities), and (provided condition a) is fulfilled) b) that the configurations (1) and (2) be u.e.*

Condition b) means the existence of a unitary operator A on V, carrying simultaneously all components U_{ij} onto the corresponding compo-

elements T_{ij}. In what follows we shall deal with u.e. configurations in which $r_j = s_j$, $j = 1, 2, \dots, m$.

4. Invariants for u.e. configurations (1) and (2), along with the dimensions of the corresponding components, are the "angles" between pairs of corresponding components (U_{ij}, U_{lk}) and (T_{ij}, T_{ik}).

By the angle $\angle(U, W)$ of subspaces U and W of the space V one means the following object (see, for example, (8)). Let π_U and π_W be the projectors of V onto U and W; π_U^W (respectively π_W^U) is the restriction of π_U (respectively π_W) to W (respectively to U).

Definition. The angle $\angle(U, W)$ between the subspaces U and W is the operator on U equal to $\pi_U^W \pi_W^U$ (first π_W^U, then π_U^W). Similarly, $\angle(W, U)$ is the operator $\pi_W^U \pi_U^W$ on W.

It turns out that $\angle(U, W)$ and $\angle(W, U)$ are nonnegative self-adjoint (and, consequently, normal!) operators on U and W. Moreover: a) for the eigenvalues λ_i of the angle $\angle(U, W)$ one has $0 \le \lambda_i \le 1$; the multiplicity of the eigenvalue 1 is equal to the dimension of $U \cap W$; the multiplicity of 0 is equal to the dimension of $U \cap W^\perp$ ($W^\perp$ is the orthogonal complement of W) and similarly for the eigenvalues λ_i' of the angle $\angle(W, U)$; b) the eigenvalues λ_i and λ_i' coincide pairwise, including multiplicities, except possibly for the multiplicities of the eigenvalue 0.

Let $d(\lambda_i)$ be the multiplicity of the eigenvalue λ_i. Renumber the λ_i in increasing order:

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_{r-1} = 1$$

and similarly for λ'_i. The expression

$$\begin{pmatrix} \lambda'_0 = 0, & \lambda_0 = 0, & \lambda_1, \dots, \lambda_{r-1} \\ d(\lambda'_0) & d(\lambda_0) & d(\lambda_1), \dots, d(\lambda_{r-1}) \end{pmatrix}$$

will be called the **metric characteristic** of the pair of subspaces U, W.

Theorem 2. *For two u.e. pairs* (U, W) *and* (U', W') *of subspaces it is necessary and sufficient that the metric characteristics of these pairs coincide.*

The proof of Theorem 2 is contained, for example, in (8).

For unitary equivalence of configurations (1) and (2), it is necessary that the metric characteristics of all possible angles $\angle(U_{ij}, U_{lk})$ from (1) coincide with the characteristics of the angles $\angle(T_{ij}, T_{lk})$ from (2) for the corresponding pairs of subspaces. But these numerous conditions, generally speaking, are not sufficient. Indeed, for the component U_{ij}, the angles $\angle(U_{ij}, U_{lk})$–for all possible U_{lk}–besides the metric characteristics, also determine orthogonal decompositions of U_{ij} into the eigenspaces of the angles $\angle(U_{ij}, U_{lk})$, and as new invariants there arise, for example, the angles between the eigenspaces of $\angle(U_{ij}, U_{lk})$ and $\angle(U_{ij}, U_{rs})$ in U_{ij}. The problem of u.e. configurations, in a certain sense, reduces to an analogous one, but for subspaces of smaller dimension. In the next section we shall carry out such a reduction in more detail.

5. The angle $\angle(U, W)$ between subspaces $U, W \subset V$ will be called **scalar** if it is a scalar linear operator λE_U (where E_U is the identity operator in U). We note that if $\angle(U, W)$ is scalar and the dimensions of the subspaces U and W are different, then $\lambda = 0$ and, consequently, U and W are orthogonal.

Suppose that in configurations (1) and (2) the metric characteristics of pairs of corresponding components coincide, and that for some pair of components U_{ij} and U_{lk} (and, consequently, also for the corresponding components T_{ij} and T_{lk}) the angles $\angle(U_{ij}, U_{lk})$ and $\angle(U_{lk}, U_{ij})$ are not simultaneously scalar. Then, decomposing in configuration (1) (and respectively in (2)) the spaces U_{ij} and U_{lk} into the orthogonal sum of eigenspaces corresponding to the given angles, we obtain two new configurations (1′) and (2′), which we shall call a **refinement of configurations** (1) and (2), or, more precisely, a refinement with respect to the indices ij and lk.

Let us note–it is not difficult to prove–that the newly arising components will form scalar angles with one another. It is also not difficult to prove that the condensed configurations (1′) and (2′) will be u.e. if and only if the original configurations (1) and (2) are u.e. Carrying out condensations successively, after a finite number of steps we arrive at configurations, say (1^*) and (2^*), such that the angles between any two components are scalar. The configurations (1^*) and (2^*) will be called **reduced**. The passage to reduced configurations is not uniquely determined, since, generally speaking, it will depend on which pairs of indices are condensed at each step. But it can be made consistent.

Theorem 3. *Configurations* (1) *and* (2) *are u.e. if and only if:*
a) all metric characteristics between the corresponding pairs of components coincide at every step of condensation, and b) the reduced configurations (1^*) *and* (2^*) *are u.e.*

Suppose that (1) and (2) are already reduced configurations. It turns out that in reduced configurations the following simple situation occurs.

Theorem 4. *Let, for every* $d \leq n$, V_{dj} *be the orthogonal sum of all components of dimension* d *in the* j*-th row of the configuration; then*

$$V_{d1} = V_{d2} = \cdots = V_{dm}.$$

Theorems 3 and 4 reduce the problem of u.e. of reduced configurations to the u.e. of configurations of the form

$$\begin{aligned} V &= U_{11} \perp U_{12} \perp \cdots \perp U_{1k}, \\ V &= U_{21} \perp U_{22} \perp \cdots \perp U_{2k}, \\ &\ldots\ldots\ldots\ldots\ldots \\ V &= U_{m1} \perp U_{m2} \perp \cdots \perp U_{mk}, \end{aligned} \tag{3}$$

$$\begin{aligned} V &= T_{11} \perp T_{12} \perp \cdots \perp T_{1k}, \\ V &= T_{21} \perp T_{22} \perp \cdots \perp T_{2k}, \\ &\ldots\ldots\ldots\ldots\ldots \\ V &= T_{m1} \perp T_{m2} \perp \cdots \perp T_{mk}, \end{aligned} \tag{4}$$

where all components U_{ik} and T_{ik} have one and the same dimension d; $n = dk$; all angles between components are scalar. Configurations of this kind will be called **special**.

It turns out that for the u.e. of two special configurations (3) and (4) it is necessary and sufficient that: a) the scalar angles between pairs of corresponding components coincide:

$$\angle(U_{ij}, U_{lk}) = \angle(T_{ij}, T_{lk});$$

b) there be u.e. of two subsequently defined sequences of unitary matrices of order $d < n$. Thus, finally, the u.e. of a sequence of matrices of order n is reduced to an analogous problem for matrices of order $d < n$. For $d = 1$, the corresponding sequences are sets of complex numbers of modulus equal to 1. For u.e. they must coincide.

In the following two sections we deal with special configurations (3) and (4).

6. Two components U_{ij} and U_{lk} will be called **adjacent**, and the adjacency relation will be denoted by $U_{ij} C U_{lk}$, if and only if

$$\angle(U_{ij}, U_{lk}) \neq 0.$$

By S we shall denote the transitive closure of the relation C: $U_{ij} S U_{lk}$ if and only if there exists a sequence of pairs of indices

$$ij = i_0 j_0,\ i_1 j_1, \dots,\ i_{rj} r = lk,$$

such that

$$\Psi_{ij,lk}: \quad U_{ij} C U_{i_1 j_1} \wedge U_{i_1 j_1} C U_{i_2 j_2} \wedge \dots \wedge U_{i_{r-1} j_{r-1}} C U_{lk}; \tag{5}$$

$\Psi_{ij,lk}$ will be called a **path** from U_{ij} to U_{lk}. We can and shall restrict ourselves only to such paths in which no pair occurs more than once. There are only finitely many such paths in the configuration.

S is an equivalence relation, and if the number p of equivalence classes with respect to S is $p > 1$, then the rows of the configurations split into p orthogonal blocks of components from one class, such that any two components from different blocks (different rows) are orthogonal. For unitary equivalence of configurations (3) and (4) it is necessary and sufficient that: a) the scalar angles coincide; b) the decomposition into blocks be the same; c) there be unitary equivalence of the configurations consisting of components belonging to the same equivalence class with respect to S.

7. If $p = 1$, then the special configuration is called **connected**. We shall indicate necessary and sufficient conditions for unitary equivalence of connected special configurations. To every path $\Psi_{ij,lk}$ from U_{ij} to U_{lk} (see (5)) we assign the following operator from U_{ij} to U_{lk}:

$$\Pi(\Psi_{ij,lk}) = \frac{1}{\sqrt{\lambda(ii_1, i_1 j_1) \cdots \lambda(i_{r-1} j_{r-1}, lk)}} \pi^{U_{lk}}_{U_{i_{r-1} j_{r-1}}} \cdots \pi^{U_{i_1 j_1}}_{U_{ij}}, \tag{6}$$

where $\lambda(mn, qt) E_{mn} = \angle(U_{mn}, U_{qt})$, and the square root is positive. $\Pi(\Psi_{ij,lk})$ carries every orthonormal basis in U_{ij} into a uniquely determined orthonormal basis in U_{lk}.

Now choose in U_{11} some basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$, and let, for some U_{lk}, $lk \neq 11$, $\Psi_{ij,lk}$ run through a nonempty (in view of connectedness!) and finite set of paths from U_{11} to U_{lk}. If $\Psi'_{11,lk}$ and $\Psi''_{11,lk}$ are two such paths, then between the bases $\Pi(\Psi'_{11,lk})(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d)$ and $\Pi(\Psi''_{11,lk})(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d)$ there is a certain unitary transition matrix of order d. We denote it by $(\Psi'_{11,lk} \to \Psi''_{11,lk})(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d)$. Generally speaking (examples show that this is indeed so), it depends on the initial basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$, but if one passes to another orthonormal basis in U_{11} by means of some transition matrix L, then $(\Psi'_{11,lk} \to \Psi''_{11,lk})(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d)$ is transformed by means of L. From this it is already not difficult to obtain the final result.

Theorem 5. For unitary equivalence of connected special configurations (3) and (4), it is necessary and sufficient that the following conditions be satisfied: a) coincidence of the scalar angles between pairs of corresponding components;

b) for arbitrarily chosen orthonormal bases $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ in U_{11} and $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_d$ in T_{11}, and for all $lk \neq 11$, simultaneous unitary equivalence of the transition matrices
$(\Psi'_{11,lk} \to \Psi''_{11,lk})(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d)$ (for (3)) with the corresponding matrices
$(\Phi'_{11,lk} \to \Phi''_{11,lk})(\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_d)$ (for (4)) for all possible pairs of paths from U_{11} to U_{lk} and, respectively, from T_{11} to T_{lk}.

Remark 1. The obtained systems of invariants for unitary equivalence are highly redundant.

Remark 2. The reduction in item 5 (coalescing) and the choice of U_{11} and T_{11} in item 7 as the initial point of the paths depend on the arbitrariness of the numbering of the components.

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