EXISTENCE THEOREMS AND CONVERGENCE OF MINIMIZING SEQUENCES FOR EXTREMUM PROBLEMS IN THE PRESENCE OF CONSTRAINTS

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Abstract

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EXISTENCE THEOREMS AND CONVERGENCE OF MINIMIZING SEQUENCES FOR EXTREMUM PROBLEMS IN THE PRESENCE OF CONSTRAINTS

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All functionals considered below are assumed to be real-valued and lower semicontinuous, and the spaces are real Banach spaces.

A functional $f(x)$ is called **quasiconvex** if

$$f\left[\frac{x+y}{2}\right] \le \max\{f(x), f(y)\}$$

for all x, y (respectively, **strictly quasiconvex** if the inequality is strict). Quasiconvexity is equivalent to convexity of $S_\lambda = \{x : f(x) \le \lambda\}$ for all λ.

Theorem 1. *A quasiconvex functional is weakly lower semicontinuous.*

Proof. Let the sequence x_n converge weakly to x. Choose a subsequence x_{n_i} for which

$$\lim_{i\to\infty} f(x_{n_i}) = \varliminf_{n\to\infty} f(x_n).$$

Then (([1], p. 457)) there is a convex combination of the points x_{n_i} converging strongly to x, i.e. there exist

$$y_{mk} = \sum_{i=k}^{k+m} \lambda_{imk} x_{n_i}, \quad \lambda_{imk} \ge 0, \quad \sum_{i=k}^{k+m} \lambda_{imk} = 1,$$

such that

$$\lim_{m\to\infty} \|y_{mk} - x\| = 0.$$

From the condition of quasiconvexity it follows that

$$f(y_{mk}) \leq \max_{k \leq i \leq k+m} \{f(x_{n_i})\};$$

therefore, by the lower semicontinuity of $f(x)$, we obtain

$$f(x) \leq \lim_{m \to \infty} f(y_{mk}) \leq \sup_{i \geq k} \{f(x_{n_i})\}.$$

Hence, as $k \to \infty$, we have

$$f(x) \leq \lim_{i \to \infty} f(x_{n_i}) = \varliminf_{n \to \infty} f(x_n).$$

Corollary. *A quasiconvex functional attains its minimum on every convex closed bounded set of a reflexive space; moreover, from every minimizing sequence one can extract a weakly convergent subsequence.*

For the case of a convex differentiable functional this result was known (([2], p. 102)).

Theorem 2. *In order that a functional attain a unique minimum on every convex closed bounded set of a reflexive space, it is necessary and sufficient that it be strictly quasiconvex. In this case every minimizing sequence converges weakly to the point of minimum.*

Theorem 3. *If there exists a functional $f(x)$ which: a) attains a minimum on every convex closed bounded set; b) satisfies the Lipschitz condition on every bounded set; c) every set of the form $\{x : f(x) \leq f(x_0)\}$ is bounded and convex, then the space E is reflexive.*

Proof. Consider

$$S = \{x : f(x) \leq \lambda\}, \quad \lambda > \inf_{x \in E} f(x).$$

It can be shown that every $c \in E^*$ attains its minimum on S, namely at the point at which the minimum of $f(x)$ is attained on the hyperplane

$$(c, x) = \inf_{x \in S} (c, x).$$

Therefore c attains its minimum also on $Q = S - S$. But Q

bounded, centrally symmetric, convex, and 0 is an interior point of Q; therefore, by means of the Minkowski functional for Q, one can define on E a norm equivalent to the original one. Applying James' s theorem ([3]) on the reflexivity of a space in which every linear functional attains its minimum on the unit ball, we obtain the required assertion.

Theorems 2 and 3 show that the requirement of quasiconvexity of the functionals and of reflexivity of the space is natural for extremum problems.

We shall now give conditions guaranteeing strong convergence of a minimizing sequence. We shall call a functional $f(x)$ **uniformly quasiconvex** if

$$f[(x+y)/2] \leq \max\{f(x), f(y)\} - \delta(\|x-y\|),$$

where $\delta(\tau)$ is a real function, $\delta(0)=0$, $\delta(\tau)>0$ for $\tau>0$.

Theorem 4. *If a uniformly quasiconvex functional is bounded below on a convex closed set Q, then it attains on Q a unique minimum, to which every minimizing sequence converges.*

We shall now give one condition, stronger but more easily checked than uniform quasiconvexity. A functional $f(x)$ is called **strongly convex** if there exists $\gamma>0$ such that

$$f[(x+y)/2] \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - \frac{1}{4}\gamma\|x-y\|^2$$

for all x, y.

Lemma. *Each of the following conditions is equivalent to strong convexity:*

a)
$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \gamma\alpha(1-\alpha)\|x-y\|^2,$$
$$0 \leq \alpha \leq 1;$$

b) *for continuous $f(x)$ there exists $c = c(x) \in E^*$ such that*

$$f(x+y) \geq f(x) + (c, y) + \gamma\|y\|^2;$$

c) *for differentiable $f(x)$*

$$f(x+y) \geq f(x) + (f'(x), y) + \gamma\|y\|^2;$$

d) *for differentiable $f(x)$*

$$(f'(x+y) - f'(x), y) \geq 2\gamma\|y\|^2;$$

e) *for twice differentiable $f(x)$*

$$(f''(x)y, y) \geq 2\gamma\|y\|^2.$$

Here everywhere $\gamma > 0$ is some constant; x, y are arbitrary.

Theorem 5. *If $f(x)$ is strongly convex, then $f(x)$, bounded below on E, and $S = \{x : f(x) \leq \lambda\}$, is bounded for every λ. If Q is convex and closed, then there exists a unique point of minimum x^*:*

$$f(x^*) = \inf_{x \in Q} f(x),$$

to which every minimizing sequence converges; moreover

$$f(x) - f(x^*) \geq \gamma \|x - x^*\|^2$$

for all $x \in Q$.

Theorem 6. *Strongly convex functionals satisfying the Lipschitz condition on every bounded set exist only in a reflexive space, and twice differentiable strongly convex functionals exist only in a Hilbert space.*

Theorems analogous to those given can be obtained by imposing additional restrictions not on the functional $f(x)$, but on the set Q. We shall call a set Q **uniformly convex** if there exists a real function $\delta(\tau)$, $\delta(0) = 0$, $\delta(\tau) > 0$ for $\tau > 0$, such that

$$\frac{1}{2}(x + y) + z \in Q$$

for any $x, y \in Q$ and any z, $\|z\| \leq \delta(\|x - y\|)$. For example, if $f(x)$ is uniformly quasiconvex and satisfies the Lipschitz condition on

$$S = \{x : f(x) \leq \lambda\},$$

then S is uniformly convex.

Theorem 7. *Let $Q \subset E$ be a bounded closed set, $f(x)$ a continuous convex functional, all supporting functionals of which on Q are nonzero. Then:*

a) *if Q is strictly convex and E is reflexive, then $f(x)$ attains a unique minimum x^* on Q, to which every minimizing sequence converges weakly;*

b) *if Q is uniformly convex and contains more than one point, then E is reflexive, and every minimizing sequence converges to x^*; moreover*

$$f(x) - f(x^*) \geq \lambda \delta(\|x - x^*\|), \qquad \lambda > 0,$$

for all $x \in Q$.

The theorems proved have many applications to problems of best approximation, mathematical programming, and optimal control. We shall consider only one example—the simplest problem of the calculus of variations in the presence of constraints ((4), Ch. VI). It is required to minimize the functional

$$f(u) = \int_0^T F(x, u, t)\, dt,$$

where

$$x(t) = x(0) + \int_0^t u(t)\,dt,$$

$x(0)$, $x(T)$, T are fixed, in the class of functions $u(t) \in L_2(0,T)$ in the presence of constraints. The constraints may be, for example, of the following form: a) $a(t) \le u(t) \le b(t)$ for almost all $t \in (0,T)$;

$$\text{b)} \int_0^T a(t)u(t)\,dt \le c; \qquad \text{c)} \int_0^T a(t)u^2(t)\,dt \le c, \quad a(t) \ge \varepsilon > 0;$$

d) $c_1 \le x(\bar{t}) \le c_2$, $\bar{t}$—a fixed point of $[0,T]$; e) $a(t) \le x(t) \le b(t)$ for all $t \in [0,T]$. Everywhere in what follows we shall assume the existence of some $u^0(t)$ satisfying all the constraints, and the continuity of $F(x,u,t)$ on $R' \times R' \times [0,T]$.

Theorem 8. *If $F(x,u,t)$ is convex in u and an arbitrary number of constraints of types a)–e) is prescribed and either 1) there are constraints of type a) or c), with in a) $a(t), b(t) \in L_2$, or 2)*

$$F(x,u,t) \ge \alpha(t)u^2(t) + \beta(t)x^2(t) + \gamma(t),$$

where $\gamma(t) \in L_1$, $\alpha(t) \ge \alpha > 0$, $z(t) \ne 0$ $(0 < t \le T)$, where $z(t)$ is the solution of the equation $(\alpha z')' - \beta z = 0$, $z(0) = 0$, $z'(0) = 1$, then a solution of the variational problem exists.

This follows from the weak lower semicontinuity of $f(u)$ and the weak compactness of the admissible set of $u(t)$.

Theorem 9. *If $F(x,u,t)$ is strictly convex in u, continuously differentiable in u and x; the (unique) constraint has the form $a(t) \le u(t) \le b(t)$, where $a(t), b(t)$ are continuous, then for every solution $u^*(t)$ there exists a continuous function $\tilde{u}(t)$ coinciding with $u^*(t)$ almost everywhere.*

The proof follows directly from Pontryagin' s maximum principle (5). We note that under the conditions of Theorems 8 and 9 one cannot assert either uniqueness of the solution or convexity of $f(u)$.

Theorem 10. *Suppose that either of the following two conditions is fulfilled:*

1. *For any $x_1, x_2, u_1, u_2 \in R'$, $t \in [0,T]$,*

$$F(x_1,u_1,t) + F(x_2,u_2,t) - 2F((x_1+x_2)/2, (u_1+u_2)/2, t) \ge$$

$$\ge \alpha(t)(u_1-u_2)^2 + \beta(t)(x_1-x_2)^2,$$

where $\alpha(t), \beta(t)$ *are the same as in Theorem 8.*

2. $F(x, u, t)$ *is twice continuously (with respect to* u *uniformly in* t*) differentiable with respect to* x *and* u*; there exist* $\gamma > \delta > 0$ *such that* $F_{uu} \geq \gamma$, $z(t) \neq 0$ *for* $0 < t \leq T$, *where* $z(t)$ *is the solution of the equation*

$$((F_{uu} - \delta)z')' - (F_{xx} - \frac{dF_{xx'}}{dt})z = 0, \quad z(0) = 0, \quad z'(0) = 1$$

(the uniform Legendre and Jacobi condition).

Then the solution $u^*(t)$ *of the problem of minimizing* $f(u)$*, for any set of constraints a)—e), exists and is unique, and every minimizing sequence* $u_n(t)$ *converges to* $u^*(t)$ *in* L_2.

Indeed, under conditions 1 or 2 the functional $f(u)$ turns out to be strongly convex, and Theorem 5 is applicable. We also note that from the convexity of $f(u)$ and of the admissible set u there follows the sufficiency of the necessary extremum conditions of the type of the maximum principle.

If T is not assumed fixed, then the above results, generally speaking, are inapplicable (since a minimizing sequence with $T \to \infty$ is possible).

Theorem 11. If $F(x, u, t) \geq \varepsilon > 0$ for all x, $u \in R'$, $t \geq 0$; $u(t) \equiv 0$ satisfies constraints of type a), then Theorems 8-10 are applicable to problems with unfixed T (if in the brachistochrone conditions $T = f(u^0)/\varepsilon$).

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Note: Figure translations are in progress. See original paper for figures.