ON A CERTAIN GENERALIZATION OF THE KINETIC THEORY OF GASES

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Abstract

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ON A CERTAIN GENERALIZATION OF THE KINETIC THEORY OF GASES

In the fundamental investigations of N. N. Bogoliubov ([2]), the kinetic equation for rarefied gases was derived from the laws of mechanics and statistics with fewer assumptions than in Boltzmann' s works ([1]). This equation has a more general form and differs from Boltzmann' s equations by additional terms, which vanish in spatially homogeneous flows. In the author' s works ([3,4]), under different and still somewhat more general assumptions, a kinetic equation was likewise obtained in generalized form, with additional corrections for the inhomogeneity of the flow. A further generalization of the kinetic theory of gases is also possible, in order to extend the range of its applicability to more complex phenomena of modern aerodynamics. In particular, we note that the kinetic theory of gases and Boltzmann' s statistical method are based on the assumption that the phase density in μ-space is proportional to the total number of particles in the system or to the mean density of the medium. This assumption lacks proper justification and is approximate in character.

In the present work the legitimacy of applying this assumption is discussed, and the possibility of a further generalization of the kinetic theory of gases is considered. For the general case of gas flow, a chain of kinetic equations is obtained in the work. To solve this chain, the method set forth in ([3,4]) is applied, and a system of kinetic equations for one-particle distribution functions is obtained.

If N identical monatomic molecules are located in a volume V, then the probability distribution function of the dynamical states of the given system is uniquely determined by the Liouville equation

$$\partial F_N/\partial t = H_N F_N \tag{1}$$

and by the initial conditions

$$F_N(0) = F_N(0, x_1, x_2, \ldots, x_N). \tag{2}$$

Instead of specifying the initial conditions exactly (the coordinates and momenta of all molecules) in the form of a δ-function in the total phase space, the

initial conditions are given a probabilistic character and are prescribed in the form of a certain distribution function smeared over phase space. If this distribution function has a small dispersion and differs little from the corresponding δ-function, then it will be asymmetric with respect to the dynamical coordinates and will in principle make it possible to follow the motion of individual particles, distinguishing identical particles by their state at the time $t = 0$. If the initial distribution function has a sufficiently large dispersion, leading to an overlap of individual states, then the possibility of distinguishing identical particles by their state disappears, and one can speak only of the behavior of the system as a whole. In this case the initial distribution function may acquire symmetry with respect to separate groups of dynamical coordinates, and in special cases with respect to all coordinates.

Thus, depending on the conditions of the problem, the Liouville equation will be satisfied by both symmetric and asymmetric distribution functions. The asymmetric distribution functions $F_N(t)$ will be expressed by $N!$ solutions. Linear combinations of such functions

will also be solutions. Therefore, from nonsymmetric functions one can construct new symmetric solutions

$$F_N^*(t) = \sum P_{kls} F_N(t), \tag{3}$$

where P_{kls} is the coordinate-permutation operator.

The distribution function is usually normalized as follows:

$$\int \cdots \int F_N(t, x_1, \ldots, x_N)\, dx_1 \ldots dx_N = 1. \tag{4}$$

We define the distribution function determining the dynamical state of the first s molecules by the expressions

$$F_s(t, x_1, \ldots, x_s) = V^s \int \cdots \int F_N\, dx_{s+1} \ldots dx_N. \tag{5}$$

The distribution functions determining the dynamical state of the first two molecules, or of only the first molecule, will be, respectively,

$$F_{12}(t, x_1, x_2) = V^2 \int \cdots \int F_N\, dx_3 \ldots dx_N; \qquad F_1(t, x_1) = V \int \cdots \int F_N\, dx_2 \ldots dx_N.$$

In the general case we shall have $N(N-1)$ two-particle distribution functions $F_{kn}(t, x_1, x_2)$ and N one-particle distribution functions $F_k(t, x_1)$. From (4) and (5) it follows that

$$\frac{1}{V}\iint F_k(t,x_1)\,dx_1 = 1.$$

Consequently, the distribution function $F_k(t,x_1)$ gives the probability-density distribution for the k-th molecule, or for the k-th state of the system.

For many dynamical systems, not each molecule but large groups of molecules will be described by their own distribution function. At a given point of phase space, probability densities may also be produced by other molecules, or by other states of the system. Therefore, in the general case

$$F_1^*(t,x_1) = \sum_k F_k(t,x_1); \qquad F_2^*(t,x_1,x_2) = \sum_k \sum_s F_{ks}(t,x_1,x_2), \ldots \quad (6)$$

If the function F_N is symmetric with respect to the permutation of any pair of coordinates, all molecules will be described by a single distribution function, and expression (6) will take the form

$$F_1(t,x_1) = N\overline{F}_1(t,x_1); \qquad F_2(t,x_1,x_2) = N(N-1)\overline{F}_2(t,x_1,x_2); \ldots \quad (7)$$

In this particular case we obtain the relations that underlie Boltzmann' s statistical method.

Let us integrate equation (1) first over all variables except x_k, then over all variables except x_k and x_r, etc.

$$\frac{\partial F_k(t,x_1)}{\partial t} = H_1 F_k + \varepsilon \sum_{l=1}^{N}{}' \iint W_2 F_{kl}(t,x_1,x_2)\,dx_2, \quad (8)$$

$$\frac{\partial F_{kr}(t,x_1,x_2)}{\partial t} = H_2 F_{kr} + \varepsilon \sum_{l=1}^{N}{}'' \iint W_3 F_{krl}(t,x_1,x_2,x_3)\,dx_3,$$

……

where ε is an auxiliary small parameter, and the operator

$$W_s = \sum_{k=1}^{s} u_{ks+1}.$$

The chain of equations (8) obtained here, in its general properties, is completely analogous to the usual chain of kinetic equations and can easily be reduced to it. Therefore, for solving (8) we shall apply the method set forth in works ([3],[4]). In order not to complicate the calculations, we shall seek solutions only for the first

two or three correlation functions. If necessary, the corresponding calculations can be made in the general case, based on the results of works ([3],[4]). For solving in (8) we set:

$$F_{krs\ldots l} = F^{(0)}_{krs\ldots l} + \varepsilon F^{(1)}_{krs\ldots l} + \varepsilon^2 F^{(2)}_{krs\ldots l} + \cdots \tag{9}$$

In addition, we take:

$$F_{kr}(t) = F_{kr}(t, x_1, x_2, \omega_k(t, x_1), \omega_r(t, x_2)), \tag{10}$$

$$F_{krs}(t) = F_{krs}(t, x_1, x_2, x_3, \omega_k(t, x_1)\omega_r(t, x_2)\omega_s(t, x_3)),$$

$$\ldots$$

We impose on the functions $\omega_l(t, x_n)$ the conditions:

$$\partial\omega_l/\partial t = \varepsilon A_l + \varepsilon^2 B_l + \varepsilon^3 C_l + \cdots, \tag{11}$$

where $A_l, B_l, \ldots$ are, for the time being, likewise unknown functions. Substituting (9), (10), (11) into (8) and collecting terms with equal powers of ε, we obtain the following systems of equations: instead of the second equation (8),

$$\left(\frac{\partial F^{(0)}_{kr}}{\partial t}\right)_\omega = H_2 F^{(0)}_{kr},$$

$$\left(\frac{\partial F^{(1)}_{kr}}{\partial t}\right)_\omega = H_2 F^{(1)}_{kr} - \left(\frac{\partial F^{(0)}_{kr}}{\partial \omega_k} A_k + \frac{\partial F^{(0)}_{kr}}{\partial \omega_r} A_r\right) + \sum_{l=1}^{N} \iint W_3 F^{(0)}_{krl}\, dx_3, \tag{12}$$

$$\ldots$$

instead of the third equation (8)

$$\left(\frac{\partial F^{(0)}_{krs}}{\partial t}\right)_\omega = H_3 F^{(0)}_{krs}, \tag{13}$$

$$\ldots$$

The auxiliary functions $\omega_k, A_k, B_k, \ldots$ are determined from equation (3)

$$\frac{\partial \omega_k(t, r_m, v_m)}{\partial t} = \left(\frac{\partial F_k}{\partial t} + v_m \frac{\partial F_k}{\partial \xi}\right)_{\xi = r_m + v_{mt}} = \varepsilon \sum_{l=1}^{N} \iint \underbrace{W_2 F_{kl}(t, \xi, v_m, x_2)}_{\xi = r_m + v_{mt}}\, dx_2. \tag{14}$$

From expressions (11) and (14) we find

$$\omega_k(t, r_m, v_m) = F_k(t, r_m + v_{mt}, v_m). \tag{15}$$

In addition, from these expressions we shall have:

$$A_k(r_m, v_m, t) = \sum_{l=1}^{N} \iint \underbrace{W_2(\xi, v_m, x_2) F_{kl}^{(0)}(t, \xi, v_m, x_2)}_{\xi = r_m + v_{mt}} \, dx_2 \tag{16}$$

...

The functions $\omega_k(t)$ are slowly varying functions of time, and in the zero approximation they do not depend on time (4). Therefore there exists such a time interval $0 \leq t \leq \tau_1$ on which they may be regarded as constant. Their dependence on time will be significant only for $t > \tau_1$.

In the zero approximation $F_{kr}^{(0)}(t)$ is determined from the first equation (12). Under the initial conditions

$$F_{kr}^{(0)}(0) = F_{kr}^{(0)}(0, x_1, x_2, \omega_k, \omega_r) \tag{17}$$

the solution can be written in the form

$$F_{kr}^{(0)}(t) = e^{tH_2} F_{kr}^{(0)}(0) = F_{kr}(0, X_1, X_2, \omega_k(X_1 t), \omega_r(X_2 t)). \tag{18}$$

In the interval $0 \leq t \leq \tau_1$ the zero approximation will be a fully reversible solution of the first equation (18). In accordance with the results of works ([3],[4]), in this time interval all subsequent approximations coincide

with the completely reversible solution (19). For $t > \tau_1$ the functions $F^{(0)}(t)$ will vary in time in the general case according to irreversible laws, leading to multiplication of the distribution functions. For $t > \tau_1$

$$F_{kr}^{(0)}(t) = F_k(Q_1^{(2)}, P_1^{(2)}, t) F_r(Q_2^{(2)}, P_2^{(2)}, t), \tag{19}$$

where

$$Q_s^{(2)} = R_s^{(2)} + \frac{t}{m} P_s^{(2)}; \qquad R_s^{(2)} = e^{tH_2} r_s; \qquad P_s^{(2)} = e^{tH_2} p_s.$$

Let us proceed to the determination of the distribution function in the first approximation $F_{kr}^{(1)}(t)$. We write the inhomogeneous term in the second equation (12) as

$$\psi_{kr}^{(1)} = \sum_{l=1}^{N}{}'' \iint \{W_3 F_{kF_{rF}} s - \widetilde{W}_3 \widetilde{F}_k \widetilde{F}_r \widetilde{F}_s\} \, dx_3, \tag{20}$$

where $\sim$ means that, in the corresponding places, the coordinates r_1 and r_2 must be replaced respectively by $r_1 + v_1 t;\ r_2 + v_2 t$. From the second equation

(12) we find the first-approximation distribution function $F_{kr}^{(1)}(t)$, satisfying zero initial conditions,

$$F_{kr}^{(1)}(t) = \int_{\tau_2}^{t} e^{(t-\tau)H_2} \psi_{kr}^{(1)}(\tau)\, d\tau. \tag{21}$$

In a completely analogous way, the following approximations and solutions for distribution functions of higher orders can be written.

Consequently, in the general case the solution of the chain of kinetic equations (8) can be written in the form:

$$F_{kr}(t) = F_k(Q_1^{(2)}, P_1^{(2)}, t) F_r(Q_2^{(2)} P_2^{(2)} t) + \varepsilon \int_{\tau_1}^{t} e^{(t-\tau)H_2} \psi_{kr}^{(1)}\, d\tau + O(\varepsilon^2), \tag{22}$$

$$F_{krs}(t) = F_k(Q_1^{(3)}, P_1^{(3)}, t) F_r(Q_2^{(3)}, P_2^{(3)}, t) F_s(Q_3^{(3)}, P_3^{(3)}, t) + \varepsilon \int_{\tau_1}^{t} e^{(t-\tau)H_2} \psi_{krs}^{(1)}\, d\tau + \ldots$$

……

The one-particle distribution functions $F_k(t, x_1)$ are determined from the following system of kinetic equations:

$$\frac{\partial F_k(t, x_1)}{\partial t} + v_1 \frac{\partial F_k}{\partial r_1} = \varepsilon \sum_{l=1}^{N} \iint W_2 F_k(1) F_l(2)\, dx_2 + O(\varepsilon^2). \tag{23}$$

As is seen, the distribution functions of the second and higher orders are expressed in terms of the one-particle distribution functions $F_k(t, x_1)$, which satisfy a certain system of kinetic equations (23).

In contrast to the classical Boltzmann kinetic theory, a system of kinetic equations (23) has been obtained from which, under certain conditions, the Boltzmann kinetic equation can be obtained approximately. The study of the solutions of the obtained system of equations (23) and their comparison with the solutions of the classical Boltzmann kinetic theory will be carried out in a separate paper.

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CITED LITERATURE

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4. V. Struminsky, DAN, 169, No. 1 (1966).

Note: Figure translations are in progress. See original paper for figures.