ON THE ASYMPTOTICS OF SOLUTIONS OF LAPLACE' S TIDAL EQUATION

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ON THE ASYMPTOTICS OF SOLUTIONS OF LAPLACE' S TIDAL EQUATION

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Free and forced oscillations of the atmosphere and ocean on a rotating planet are described by Laplace' s tidal equation

$$\left[(1-\mu^2)\frac{d}{d\mu}-\frac{s\mu}{f}\right]\psi=(f^2-\mu^2)\xi,$$

$$\left[(1-\mu^2)\frac{d}{d\mu}+\frac{s\mu}{f}\right]\xi=\left[\frac{s^2}{f^2}-(1-\mu^2)\gamma\right]\psi \tag{1}$$

(1-3); s is an integer; f is a dimensionless frequency, $f=\sigma/2\omega$ (σ is the frequency of oscillations, ω is the angular velocity of rotation of the planet); $\gamma=4a^2\omega^2/gh$, where h is the so-called dynamic equivalent depth. The solutions must be regular at the endpoints of the interval $[-1,1]$. Such solutions exist not for all values of f,γ. In the plane (f^{-1},γ^{-1}) a family of eigenvalue curves is formed. The asymptotics of these curves for large γ^{-1} is well known ([5]). They are divided into two groups. For curves of the first kind (we shall say "curves of the first kind on the right," i.e., for large γ^{-1}):

$$\gamma^{-1}=\frac{f^2-1}{(s/f+n)(-s/f+n+1)}+\frac{(n-s)(n+s)(s/f-n+1)}{(2n-1)(2n+1)(s/f+n)(s/f-n(n-1))}+$$

$$+\frac{(n-s+1)(n+s+1)(s/f+n+2)}{(2n+1)(2n+3)(s/f-n-1)(s/f-(n+1)(n+2))}, \tag{2}$$

where n is the number of the curve, $n=s,s+1,\ldots.$ For curves of the second kind on the right

$$\frac{s}{f}\sim n(n+1)+\frac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)\left[(n-1)^2n^2\gamma^{-1}+2s^2n^{-1}(n+1)^{-2}\right]}+$$

$$+\frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)\left[(n+1)^2(n+2)^2\gamma^{-1}-2s^2n^{-2}(n+1)^{-1}\right]}, \tag{3}$$

$n = s, s+1, \dots$. Curves of the first kind describe fast, high-frequency waves; curves of the second kind, slow inertial waves of gyroscopic origin. In the present work the asymptotics on the left, for small γ^{-1}, is found. This asymptotics is important in the study of internal waves with large vertical wave numbers, and also in the study of waves on rapidly rotating planets ([8]). The curves are also divided into two groups. For curves of the first kind on the left, for small γ^{-1}, we have

$$f^2\gamma^{1/2} \sim (2p+1) - sf/(2p+1). \tag{4}$$

For curves of the second kind on the left

$$f\gamma^{1/2} \sim s/(2p+1), \tag{5}$$

where p is some integer. In order to determine how the asymptotics on the right (2)–(3) joins with the asymptotics on the left (4)–(5), it is necessary to compute the eigenvalue curves in the intermediate region.

Figure 1 shows the results of a computation by the method described in ([3]). Dashed lines depict the asymptotics for large γ^{-1}, dash-dotted lines for small γ^{-1}, and solid lines the exact curves. The case $s = 1$ is presented. The cases $s = 2$ and $s = 3$ were also computed; there the picture turns out to be analogous. The lower of the curves of the second kind on the right (according to

in formula (3) turns out to be the upper one of the curves of the first kind on the left (formula (4)). For the remaining curves the kind on the left and on the right coincides. The asymptotic formula gives a good result over almost the entire extent of the curves.

Let us outline the proof of formulas (4)–(5). By differentiating (1) one can obtain the equation
$(L + s/f)\xi = \gamma[(s/f+2)\mu - (1-\mu^2)d/d\mu]\psi$, where

$$L = \frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} - \frac{s^2}{1-\mu^2}$$

(see (3)). We are interested in the asymptotics as $\gamma^{-1} \to 0$, $f \to 0$, and therefore in the right-hand side the 2 may be omitted in comparison with s/f. Taking (1) into account, we shall have the equation
$(L + s/f)\xi = -\gamma(f^2 - \mu^2)\xi$. If we introduce $E = f^2 + s/f\gamma$, then we obtain

Fig. 1

Fig. 1

Figure 1: Fig. 1

$$-\frac{d}{d\mu}(1-\mu^2)\frac{d\xi}{d\mu}+\frac{s^2}{1-\mu^2}\xi+\gamma(\mu^2-E)\xi=0. \tag{6}$$

The problem of the asymptotic behavior of the eigenvalues E for large γ is completely analogous to the quasiclassical approximation in quantum mechanics and can be solved by the WKB method (see, for example, [6]). We shall follow the so-called Zwann method (see [7]). Equation (6), unlike the Schrödinger equation, has singularities at the ends of the interval. It can be shown that approximate solutions are

$$\xi=[(1-\mu^2)p]^{-1/4}\exp\left[\pm\sqrt{\gamma}\int\sqrt{p/(1-\mu^2)}\,d\mu\right], \qquad p=\mu^2-E. \tag{7}$$

The approximation cannot hold in neighborhoods of the endpoints and of the points $\mu_0=\pm\sqrt{E}$, where (7) has branch points absent in the exact solution. We shall go around these branch points (turning points) in the complex plane. The lines issuing from the turning points on which

$$\operatorname{Re}\int_{\pm\sqrt{E}}^{\mu}\sqrt{p/(1-\mu^2)}\,d\mu=0$$

are called Stokes lines. They are shown in Fig. 2. Consider a domain containing only one turning point

(Fig. 3). The following assertion is valid: whatever the solution of equation (6), on any of the Stokes lines and in the two sectors surrounding this line this solution can be approximated by a linear combination of the functions (7). More precisely,

$$\xi=\sum_{l=1}^{2}c_l(\gamma)\left[(1-\mu^2)p\right]^{-1/4}\exp\left[\pm\sqrt{\gamma}\int_{\mu_0}^{\mu}\sqrt{p/(1-\mu^2)}\,d\mu\right]\times$$
$$\times\left[1+O\left(\gamma^{-1/2}\sqrt{E}\,p\right)^{-3/2}\right]. \tag{8}$$

In each of such regions the linear combination is determined uniquely. In the common sector of two regions the corresponding linear combinations differ from one another by the coefficient at the exponential that is decaying in this sector. For each of the Stokes lines S_k and the two surrounding sectors we introduce a standard system of functions of type (7). Namely, let on S_k there be

$$u_{1,2}^{(k)} = \frac{a^{(k)}}{\sqrt[4]{(1-\mu^2)p}} \exp\left[\pm i\sqrt{\gamma}\int_{\mu_0}^{\mu}\sqrt{p/(1-\mu^2)}\,d\mu\right],$$

where $a^{(k)}$ are chosen so that $\left|a^{(k)}\right| = 1$, $\lim_{\mu\to\mu_0,\ \mu\in S_k} \arg a^{(k)} p^{-1/4} = 0$. Suppose that some solution ξ is approximated on S_1 and in the two surrounding sectors by the linear combination $c_1^{(1)}u_1^{(1)} + c_2^{(1)}u_2^{(1)}$, and on S_2 and in the surrounding sectors by the combination $c_1^{(2)}u_1^{(2)} + c_2^{(2)}u_2^{(2)}$. How are the coefficients of these linear combinations related to one another? Introduce the vectors $c^{(1)} = (c_1^{(1)}, c_2^{(1)})$, $c^{(2)} = (c_1^{(2)}, c_2^{(2)})$. It is proved that $c^{(2)} = \Omega c^{(1)}$, where

$$\Omega = e^{-i\pi/6}\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}.$$

The same matrix effects the transition from S_2 to S_3 and from S_3 to S_1. Let us now return to Fig. 2. We seek the asymptotics in the whole domain, except for fixed neighborhoods of the endpoints and neighborhoods of the points $\mu = \pm\sqrt{E}$ of radius $\varepsilon\sqrt[3]{E}$ (ε is a fixed small number). The remainder terms in (8) will be $O(\gamma^{-1/2}E^{-3/4})$. In the region $I + II$ one has $\xi \sim c_1(\gamma)u_1^{(3)} + c_2(\gamma)u_2^{(3)}$. In particular, on the interval $[\sqrt{E}, 1]$, outside the deleted neighborhoods,

$$\xi \sim e^{i\pi/12}\left[(1-\mu^2)(\mu^2-E)\right]^{-1/4}\sum_{l=1}^{2} c_l(\gamma)\exp\left[\pm\sqrt{\gamma}\int_{\sqrt{E}}^{\mu}\sqrt{\frac{\mu^2-E}{1-\mu^2}}\,d\mu\right]. \quad (9)$$

The coefficients $c_{1,2}$ are determined by the boundary conditions at $\mu = 1$. It is easy to show that on the interval $[\sqrt{E}, 1]$ the solution must tend monotonically to zero. Suppose, for example, that at some point there is a positive maximum, $\xi' = 0$, $\xi > 0$. Equation (6) shows that at this point $\xi'' > 0$, which is impossible for a maximum. Therefore in formula (9) the coefficient of the increasing exponential, c_1, must be small in comparison with the other one, namely $c_1/c_2 \to 0$ as $\gamma \to \infty$. To obtain the asymptotics on S_1, one must apply the matrix Ω to the vector (c_1, c_2).

On the interval $[-\sqrt{E}, \sqrt{E}]$ (outside the deleted neighborhoods)

$$\xi \sim e^{-i\pi/6}\left[(1-\mu^2)(E-\mu^2)\right]^{-1/4}\sum_{l=1}^{2} b_l\exp\left[\pm i\sqrt{\gamma}\int_{\mu}^{\sqrt{E}}\sqrt{\frac{E-\mu^2}{1-\mu^2}}\,d\mu\right], \quad (10)$$

where $b_1 = c_2$, $b_2 = c_1 + ic_2$. Denote

$$A = \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{\frac{E-\mu^2}{1-\mu^2}}\, d\mu.$$

Then

$$\xi \sim e^{-i\pi/6}\left[(1-\mu^2)(E-\mu^2)\right]^{-1/4} \sum_{l=1}^{2} b_l \exp\left[\pm i\sqrt{\gamma}\int_{-\sqrt{E}}^{\mu}\sqrt{\frac{E-\mu^2}{1-\mu^2}}\, d\mu\right],$$

$$b_1 = \exp[i\sqrt{\gamma}A]c_2, \qquad b_2 = \exp[-i\sqrt{\gamma}A](c_1 + ic_2).$$

With the aid of Ω we pass from S_1 to S_4:

$$\xi \sim e^{-i\pi/3}\sum_{l=1}^{2} b_{lu} l^{(4)}, \qquad b_1 = e^{i\sqrt{\gamma}A}c_2, \qquad b_2 = (c_1 + ic_2)e^{-i\sqrt{\gamma}A} + ic_2 e^{i\sqrt{\gamma}A}.$$

In particular, on the interval $[-1, -\sqrt{E}]$,

$$\xi \sim e^{-i5\pi/12}\left[(1-\mu^2)(\mu^2-E)\right]^{-1/4}\Bigg\{c_2 e^{i\sqrt{\gamma}A}\exp\left[-\sqrt{\gamma}\int_{\mu}^{-\sqrt{E}}\sqrt{\frac{\mu^2-E}{1-\mu^2}}\, d\mu\right] + \left(c_1 e^{-i\sqrt{\gamma}A} + 2ic_2\cos\sqrt{\gamma}A\right)\exp\left[\sqrt{\gamma}\int_{\mu}^{-\sqrt{E}}\sqrt{\frac{\mu^2-E}{1-\mu^2}}\, d\mu\right]\Bigg\}.$$

The coefficient of the growing exponential is infinitely small in comparison with the coefficient of the decaying exponential. Therefore $\cos\sqrt{\gamma}A = 0$ and

$$\sqrt{\gamma}\int_{-\sqrt{E}}^{\sqrt{E}}\sqrt{\frac{E-\mu^2}{1-\mu^2}}\, d\mu \sim \pi\left(p + \frac{1}{2}\right), \tag{11}$$

where p is an integer. This condition is analogous to Bohr' s quantization rule. As $\gamma \to \infty$, evidently $E \to 0$. Therefore $1-\mu^2$ in the principal term of the asymptotics may be replaced by 1. Integrating, we obtain $\sqrt{\gamma}E = 2p+1$, or $\sqrt{\gamma}f^2 + s/f\sqrt{\gamma} = 2p+1$ (> 0). Solving with respect to $\sqrt{\gamma}$, we have two branches. Expanding in powers of f, we arrive at formulas (4) and (5). Detailed computations of the eigenperiods and eigenfunctions for $s = 1, 2, 3$ are given in (8). In paper (4) it is indicated that the singularities of the solutions for large γ possibly play a decisive role in the explanation of atmospheric tides.

Remarks. 1. Strictly speaking, this derivation of formula (11) is valid for large p, since otherwise the WKB approximation (10) on the interval $(-\sqrt{E}, \sqrt{E})$ may be unsuitable. If one is not interested in the asymptotics of the eigenfunctions, but only in the eigenvalues, then both turning points can be bypassed in the complex plane (see (7), p. 111) and (11) can be proved for all p.

2. It can be shown that the asymptotics (11) actually gives either the eigenvalues f themselves or those taken with the opposite sign, $-f$. For waves of the first kind this is immaterial, since the principal term of the asymptotics is symmetric with respect to the sign of f, i.e., to the direction of propagation of the wave. For waves of the second kind this means that if solutions with positive f existed for $2p + 1 = 3, 5, \ldots$, then the solution with negative f must be for $2p + 1 = 1$. Thus, there is one wave of the second kind with direction of propagation from west to east. Prof. Longuet-Higgins drew our attention to its existence, for which we express our gratitude to him.

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Note: Figure translations are in progress. See original paper for figures.