ON THE ALGEBRAIC INDEPENDENCE OF VALUES OF E-FUNCTIONS SATISFYING LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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Abstract

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ON THE ALGEBRAIC INDEPENDENCE OF VALUES OF E-FUNCTIONS SATISFYING LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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In the note ([1]) the author set out a method applicable to proving the algebraic independence over the field of rational functions of a solution of a linear homogeneous differential equation of the 3rd order and its derivatives. In the present note the indicated method is transferred to nonhomogeneous equations, which makes it possible to apply it in the general case to proving the algebraic independence of values of E-functions* satisfying linear differential equations of the 3rd order.

Consider the differential equation

$$y''' + p_2(z)y'' + p_1(z)y' + p_0(z)y = q(z), \tag{1}$$

whose coefficients $p_i = p_i(z),\ i = 0,\ 1,\ 2,\ q = q(z)$ are rational functions of z.

Let, further,

$$P(y'', y', y, z) = 0 \tag{2}$$

be an algebraic differential equation whose left-hand side is an irreducible polynomial in y, y', y'' with coefficients from the field of rational functions. Denote by $\overline{P} = \overline{P}(y'', y', y, z)$ the polynomial, homogeneous with respect to y, y', y'', composed of the terms of the highest total degree in y, y', y'' in P.

Theorem 1 ([2]). *Let $y(z) \neq 0$ simultaneously satisfy equations (1), (2) and not be a solution of any algebraic differential equation of order less than 2. Then there exists a solution $\overline{y}(z) \neq 0$ simultaneously satisfying the equations*

$$y''' + p_2y'' + p_1y' + p_0y = 0, \tag{3}$$

$$\overline{P}(y'', y', y, z) = 0. \tag{4}$$

We shall apply the indicated fact and the method of note ([1]) to the investigation of the arithmetic nature of the values of the E-functions

$$K_{\beta_1\beta_2\beta_3}(z) = \sum_{n=0}^{\infty} \frac{1}{(\beta_1+1)\dots(\beta_1+n)(\beta_2+1)\dots(\beta_2+n)(\beta_3+1)\dots(\beta_3+n)} \left(\frac{z}{3}\right)^{3n},$$

$$K_{\beta_1\beta_2\beta_3\alpha_1}(z) = \sum_{n=0}^{\infty} \frac{\alpha_1(\alpha_1+1)\dots(\alpha_1+n-1)}{(\beta_1+1)\dots(\beta_1+n)(\beta_2+1)\dots(\beta_2+n)(\beta_3+1)\dots(\beta_3+n)} \left(\frac{z}{2}\right)^{2n},$$

$$K_{\beta_1\beta_2\beta_3\alpha_1\alpha_2}(z) = \sum_{n=0}^{\infty} \frac{\alpha_1(\alpha_1+1)\dots(\alpha_1+n-1)\alpha_2(\alpha_2+1)\dots(\alpha_2+n-1)}{(\beta_1+1)\dots(\beta_1+n)(\beta_2+1)\dots(\beta_2+n)(\beta_3+1)\dots(\beta_3+n)} z^n,$$

* For the definition of an E-function see, for example, ([4]).

where $\beta_1,\ \beta_2,\ \beta_3 \neq -1,\ -2, \dots;\ \alpha_1,\ \alpha_2 \neq 0,\ -1,\ -2, \dots$, satisfying, respectively, the linear differential equations

$$y''' + \frac{3\sigma_1(\beta)+3}{z} y'' + \frac{9\sigma_2(\beta)+3\sigma_1(\beta)+1}{z^2} y' + \left[\frac{27\sigma_3(\beta)}{z^3} - 1\right] y = \frac{27\sigma_3(\beta)}{z^3}, \tag{5}$$

$$y''' + \frac{2\sigma_1(\beta)+3}{z} y'' + \left[\frac{4\sigma_2(\beta)+2\sigma_1(\beta)+1}{z^2} - 1\right] y' + \left[\frac{8\sigma_3(\beta)}{z^3} - \frac{2\alpha_1}{z}\right] y = \frac{8\sigma_3(\beta)}{z^3}, \tag{6}$$

$$y''' + \left[\frac{\sigma_1(\beta)+3}{z} - 1\right] y'' + \left[\frac{\sigma_2(\beta)+\sigma_1(\beta)+1}{z^2} - \frac{\sigma_1(\alpha)+1}{z}\right] y' + \left[\frac{\sigma_3(\beta)}{z^3} - \frac{\sigma_2(\alpha)}{z^2}\right] y = \frac{\sigma_3(\beta)}{z^3}, \tag{7}$$

where $\sigma_1(\beta) = \beta_1+\beta_2+\beta_3$, $\sigma_2(\beta) = \beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3$, $\sigma_3(\beta) = \beta_1\beta_2\beta_3$, $\sigma_1(\alpha) = \alpha_1+\alpha_2$, $\sigma_2(\alpha) = \alpha_1\alpha_2$.

With respect to the first of these functions, $K_{\beta_1\beta_2\beta_3}$, the following holds.

Theorem 2. *Let $\beta_1,\ \beta_2,\ \beta_3$ be rational numbers satisfying the conditions*

$$|2\beta_1-\beta_2-\beta_3| \neq n_1, \quad |2\beta_2-\beta_1-\beta_3| \neq n_2, \quad |2\beta_3-\beta_1-\beta_2| \neq n_3,$$

where $n_1,\ n_2,\ n_3$ are natural numbers.

Then the numbers

$$K_{\beta_1\beta_2\beta_3}(z),\quad K'_{\beta_1\beta_2\beta_3}(z),\quad K''_{\beta_1\beta_2\beta_3}(z)$$

are algebraically independent for any algebraic value $z \neq 0$.*

We outline the proof. By Shidlovskii' s main theorem ([4]), it is enough to prove that, for the indicated values of $\beta_1,\ \beta_2,\ \beta_3$, the functions

$$K_{\beta_1\beta_2\beta_3},\quad K'_{\beta_1\beta_2\beta_3},\quad K''_{\beta_1\beta_2\beta_3}$$

are algebraically independent over the field of rational functions of z.

Consider the function $K = z^{3\beta_3}K_{\beta_1\beta_2\beta_3}$, satisfying the equation

$$y''' + \frac{3(\lambda+\mu+1)}{z}y'' + \frac{(3\lambda+1)(3\mu+1)}{z^2}y' - y = 27\sigma_3(\beta)z^{3(\beta_3-1)},$$

where $\lambda = \beta_1 - \beta_3,\ \mu = \beta_2 - \beta_3$. To prove the theorem it is enough to show that the functions $K,\ K',\ K''$ are algebraically independent over the field of rational functions.

The function K, differing by the factor $z^{3\beta_3}$ from the entire and non-polynomial function $K_{\beta_1\beta_2\beta_3}$, is not algebraic. Therefore, if K is a solution of an algebraic differential equation of the first order, then there exists an algebraic function $f(z, z_1)$ satisfying the equation

$$G_2^2 f + \frac{3(\lambda+\mu+1)}{z}G_2 f + \frac{(3\lambda+1)(3\mu+1)}{z^2}f - z_1 - 27\sigma_3(\beta)z^{3(\beta_3-1)} = 0$$

(see ([1]), Theorem 2). Substituting into this equation the expansion

$$f(z, z_1) = h_0(z)z_1^{\varepsilon_0} + h_1(z)z_1^{\varepsilon_1} + \cdots,\qquad \varepsilon_0 > \varepsilon_1 > \cdots$$

(Theorem 5) and comparing coefficients of the highest powers of z_1, we obtain a condition on the algebraic function $h_0(z)$:

$$h^3 + 3hh' + h'' + \frac{3(\lambda+\mu+1)}{z}(h^2 + h') + \frac{(3\lambda+1)(3\mu+1)}{z^2}h - 1 = 0.$$

Investigating the expansion of the algebraic function $h_0(z)$ in a neighborhood of an arbitrary point $z = a$, one can show that $h_0(z)$ is a rational function having only simple poles. Comparing the expansion of this function into partial fractions with its expansion in a neighborhood of $z = \infty$, we obtain

$$\lambda + \mu = n_1,\quad \lambda - 2\mu = n_2,\quad \mu - 2\lambda = n_3,$$

where $n_1,\ n_2,\ n_3 = -1, -2, ...$, which contradicts the conditions of the theorem, since

$$\lambda = \beta_1 - \beta_3, \qquad \mu = \beta_2 - \beta_3.$$

* In the case $\beta_1 = \beta_2 = \beta_3$, this fact is contained in Theorem 6 of ([3]).

If now the functions K, K', K'' are algebraically dependent, then, by Theorem 1, there exists $\overline{y}(z) \not\equiv 0$ satisfying simultaneously equations (3) and (4), where $p_0 = -1,\ p_1 = (3\lambda+1)(3\mu+1)/z^2,\ p_2 = 3(\lambda+\mu+1)/z$. In this case the left-hand side of equation (3) coincides with the left-hand side of the homogeneous equation (5) considered in [1]. By Theorem 5 of [1], for values of λ, μ satisfying the conditions $|\lambda+\mu| \neq n_1,\ |\lambda-2\mu| \neq n_2,\ |\mu-2\lambda| \neq n_3,\ n_1, n_2, n_3 = 1, 2, ...$, this equation is differentially irreducible and, consequently, cannot have common solutions $\overline{y}(z) \not\equiv 0$ with equation (4). The contradiction obtained proves our theorem.

The following assertions are proved analogously:

Theorem 3. Let $\beta_1, \beta_2, \beta_3, \alpha_1$ be rational numbers satisfying the conditions

$$\alpha_1 - \beta_i \neq n_i/2, \quad i = 1, 2, 3, \quad \text{where } n_i = 0, -1, \pm 2, \pm 3, \dots;$$

$$\alpha_1 + \beta_i - \beta_j - \beta_k \neq (2n_{ijk} - 1)/2, \quad i, j, k = 1, 2, 3; \quad i \neq j \neq k; \quad n_{ijk} = 0, -1, \pm 2, \pm 3, \dots.$$

Then the numbers $K_{\beta_1\beta_2\beta_3\alpha_1}(z),\ K'_{\beta_1\beta_2\beta_3\alpha_1}(z),\ K''_{\beta_1\beta_2\beta_3\alpha_2}(z)$ are algebraically independent for any algebraic $z \neq 0$.

Theorem 4. Let $\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2$ be rational numbers satisfying the conditions

$$\alpha_i - \beta_j \neq n_{ij}, \quad i = 1, 2; \quad j = 1, 2, 3, \quad \text{where } n_{ij} = 0, \pm 1, \pm 2, \dots;$$

$$\alpha_1 + \alpha_2 - \beta_j - \beta_k \neq n_{jk}, \quad j, k = 1, 2, 3; \quad j \neq k; \quad n_{jk} = 0, -1, \pm 2, \pm 3, \dots.$$

Then the numbers $K_{\beta_1\beta_2\beta_3\alpha_1\alpha_2}(z),\ K'_{\beta_1\beta_2\beta_3\alpha_1\alpha_2}(z),\ K''_{\beta_1\beta_2\beta_3\alpha_1\alpha_2}(z)$ are algebraically independent for any algebraic $z \neq 0$.

Making in equations (6), (7), respectively, the substitutions $z^{2\beta_3}y,\ z^{\beta_3}y$ and putting $\beta_1 - \beta_3 = \lambda,\ \beta_2 - \beta_3 = \mu$, we arrive at the linear differential equations

$$y''' + \frac{2\lambda + 2\mu + 3}{z}\, y'' + \frac{(2\lambda+1)(2\mu+1) - z^2}{z^2}\, y' - \frac{2\alpha_1}{z}\, y = 8\sigma_3(\beta) z^{2\beta_3 - 3},$$

$$y''' + \frac{\lambda + \mu + 3 - z}{z}\, y'' + \frac{(\lambda+1)(\mu+1) - (\alpha_1 + \alpha_2 + 1)z}{z^2}\, y' - \frac{\alpha_1\alpha_2}{z^2}\, y = \sigma_3(\beta) z^{\beta_3 - 3},$$

whose left-hand sides coincide with the left-hand sides of the equations considered in [1].

Using the techniques indicated above, we obtain the assertions of Theorems 3 and 4.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.