THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR SECOND-ORDER HYPERBOLIC EQUATIONS DEGENERATING IN THE DOMAIN AND ON ITS BOUNDARY

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THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR SECOND-ORDER HYPERBOLIC EQUATIONS DEGENERATING IN THE DOMAIN AND ON ITS BOUNDARY

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The Cauchy problem for second-order hyperbolic equations degenerating on the manifold where the initial conditions are prescribed, as well as the boundary-value problem for such equations, have been considered in the works of many authors (see, for example, ($^{1-7}$)). In the present note, applying the methods of the papers ($^{8-10}$) on second-order equations with nonnegative characteristic form (see also (^{11})), we shall study the Cauchy problem and the boundary-value problem for second-order hyperbolic equations degenerating in an arbitrary way inside the domain under consideration and on its boundary. We shall consider the equation

$$\begin{aligned} L(u) \equiv -u_{tt} + (a^{ij}(t,x)u_{x_i})_{x_j} + b^i(t,x)u_{x_i} + b^0(t,x)u_t + \\ + c(t,x)u = f(t,x), \qquad (1) \end{aligned}$$

where $a^{ij}(t,x)\xi_i\xi_j \geqslant 0$ for all (t,x) with $t \geqslant 0$. Such equations may naturally be called **limit hyperbolic**.

1. The Cauchy problem. In the domain $G\{0 \leqslant t \leqslant T,\ x \subset R_m\}$, $R_m = (x_1, \dots, x_m)$, we shall study the Cauchy problem for equation (1) with initial conditions

$$u(0,x) = 0, \qquad u_t(0,x) = 0. \qquad (2)$$

Lemma 1. For solutions of the Cauchy problem for the equation

$$-u_{tt} + \varepsilon\Delta u + (a_\varepsilon^{ij}u_{x_i})_{x_j} + b_\varepsilon^i u_{x_i} + b_\varepsilon^0 u_t + c_\varepsilon u = f_\varepsilon, \qquad \varepsilon > 0, \qquad (3)$$

with infinitely differentiable coefficients and f_ε in G, under conditions (2), the estimates

$$\left\|\frac{\partial^k u}{\partial t^{k_0}\partial x_1^{k_1}\dots\partial x_m^{k_m}}\right\|_{L_2(G)} \leqslant M, \qquad k \geqslant 0, \tag{4}$$

hold, where M does not depend on ε, if all derivatives of the coefficients of (3) and of f_ε up to order k inclusive are uniformly bounded with respect to ε, and if there exist constants $A > 0$ and $\alpha > 0$, independent of ε, such that

$$Aa_\varepsilon^{ij}\xi_i\xi_j + a_{\varepsilon t}^{ij}\xi_i\xi_j - \alpha\bigl(b_\varepsilon^i\xi_i\bigr)^2 \geqslant 0; \tag{5}$$

f_ε are finite with respect to x. For $k = 0$, in addition, it is assumed that $a_{\varepsilon t}^i$, $b_{\varepsilon x_i}^0$, $b_{\varepsilon t}^0$ are uniformly bounded with respect to ε, and for $k = 1$ also $a_{\varepsilon x_i x_\rho}^{ij}$ and $b_{\varepsilon x_i t}^0$.

Proof. Multiply equation (3) by $e^{\theta t}w$, where

$$w = \int_t^\tau u(s,x)\,ds,$$

and integrate it over $G_\tau\{0 \leqslant t \leqslant \tau,\ x \subset R_m\}$. Since f_ε is finite, the solutions of problems (2), (3) are also finite. Consider ···

individual terms of the obtained equality. For brevity of notation put

$$[\varphi,\psi]_{G_\tau} \equiv \int_{G_\tau}\varphi\psi\,dG, \qquad (\varphi,\psi)_{t=\tau} \equiv \int_{R_m}\varphi(\tau,x)\psi(\tau,x)\,dx.$$

Integrating by parts, we obtain

$$[u_{tt}, e^{\theta t}w]_{G_\tau} = \frac{1}{2}(u, e^{\theta t}u)_{t=\tau} + [ue^{\theta t}, \theta^2 w - \tfrac{3}{2}\theta u]_{G_\tau};$$

$$[\varepsilon\Delta u, e^{\theta t}w]_{G_\tau} = -\frac{1}{2}\varepsilon\theta[w_{x_i}, e^{\theta t}w_{x_i}]_{G_\tau} - \frac{1}{2}\varepsilon(w_{x_i}, w_{x_i})_{t=0};$$

$$[(a_\varepsilon^{ij}u_{x_i})_{x_j}, e^{\theta t}w]_{G_\tau} = -\frac{1}{2}[(\theta a_\varepsilon^{ij} + a_{\varepsilon t}^{ij})w_{x_i}, e^{\theta t}w_{x_j}]_{G_\tau} - \frac{1}{2}(a_\varepsilon^{ij}w_{x_i}, w_{x_j})_{t=0}; \tag{6}$$

$$[b_\varepsilon^i u_{x_i}, e^{\theta t}w]_{G_\tau} = -[b_{\varepsilon x_i}^i u, e^{\theta t}w]_{G_\tau} - [b_\varepsilon^i w_{x_i}, e^{\theta t}u]_{G_\tau}; \tag{7}$$

$$[b^0_\varepsilon u_t, e^{\theta t} w]_{G_\tau} = [u, e^{\theta t} b^0_\varepsilon u - (b^0_\varepsilon e^{\theta t})_{tw}]_{G_\tau}.$$

Using the inequality

$$2ab \leq \alpha a^2 + \alpha^{-1} b^2 \tag{8}$$

to estimate the last integral in (7), and condition (5) to estimate the terms (6) and (7), choosing $\theta > 0$ sufficiently large, we obtain from the equality under consideration that

$$(u, u)_{t=\tau} \leq M_1[u, u]_G + M_2[f_\varepsilon, f_\varepsilon]_G. \tag{9}$$

Here M_i, here and below, do not depend on ε and τ, $0 \leq \tau \leq T$. From (9) it follows that $[u, u]_G \leq M$. Suppose that the estimates (4) are valid for $k-1$. We show that they are valid also for k. Apply to (3) the operator

$$D^k \equiv \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_m}}{\partial x_m^{k_m}},$$

multiply the resulting equation by $e^{\beta t} D^{kw}$, integrate over G_τ, and sum over all possible D^k. Transform the individual terms of this equality:

$$[D^k u_{tt}, e^{\beta t} D^{kw}]_{G_\tau} = \frac{1}{2}(D^{ku}, e^{\beta t} D^{ku})_{t=\tau} + [e^{\beta t} D^{ku}, \beta^2 D^{kw} - \tfrac{3}{2}\beta D^{ku}]_{G_\tau};$$

$$[\varepsilon D^k \Delta u, e^{\beta t} D^{kw}]_{G_\tau} = -\frac{1}{2}[D^k w_{x_i}, \varepsilon\beta e^{\beta t} D^k w_{x_i}]_{G_\tau} - \frac{1}{2}(\varepsilon D^k w_{x_i}, D^k w_{x_i})_{t=0}.$$

In what follows, by A_i^k we shall denote constants for which the estimate

$$|A_i^k| \leq M_3[D^{ku}, D^{ku}]_{G_\tau} + M_4,$$

is valid.

$$[D^k(a^{ij}_\varepsilon u_{x_i})_{x_j}, e^{\beta t} D^{kw}]_{G_\tau} = -\frac{1}{2}[(\beta a^{ij}_\varepsilon + a^{ij}_{\varepsilon t}) D^k w_{x_i}, e^{\beta t} D^k w_{x_j}]_{G_\tau}$$

$$-(a^{ij}_\varepsilon D^k w_{x_i}, D^k w_{x_j})_{t=0} - [a^{ij}_{\varepsilon x_\rho} D^{k-1} w_{x_i x_j}, e^{\beta t} D^{ku}]_{G_\tau} + A_1^k.$$

We estimate the last integral, using (8) and Lemma 1 of [9], according to which

$$(a^{ij}_{\varepsilon x_\rho} D^{k-1} w_{x_i x_j})^2 \leq M_5 a^{ij}_{\varepsilon} D^k w_{x_i} D^k w_{x_j}.$$

Further,

$$[D^k(b^i_\varepsilon u_{x_i}), e^{\beta t} D^{kw}]_{G_\tau} = A^k_2 - [b^i_\varepsilon D^k w_{x_i}, e^{\beta t} D^{ku}]_{G_\tau}; \tag{10}$$

$$[D^k(b^0_\varepsilon u_t), e^{\beta t} D^{kw}]_{G_\tau} = A^k_3 + [b^0_\varepsilon D^k u_t + b^0_{\varepsilon x_\rho} D^{k-1} u_t, e^{\beta t} D^{kw}]_{G_\tau}. \tag{11}$$

The last integral in (10) is estimated as in (7), while the integral in (11) is transformed by integration by parts with respect to t. Taking all these transformations into account, in the same way as (9), we obtain that

$$(D^{ku}, D^{ku})_{t=\tau} \leq M_6 [D^{ku}, D^{ku}]_{G_\tau} + M_7,$$

and, consequently,

$$[D^{ku}, D^{ku}]_G \leq M_8.$$

The boundedness, uniform in ε, in $\mathcal{L}_2(G)$ of the derivatives of order k of u containing more than one differentiation with respect to t, is obtained by differentiating equation (3). To estimate in $\mathcal{L}_2(G)$ derivatives of the form $D^{k-1}u_t$, we apply the operator D^{k-1} to equation (3), multiply it by $D^{k-1}u_t$, integrate over G_τ, and transform the individual terms by integration by parts. Thus, Lemma 1 is proved.

Theorem 1. Suppose that for the coefficients of equation (1) the inequality

$$A a^{ij} \xi_i \xi_j + a^{ij}_t \xi_i \xi_j - \alpha (b^i \xi_i)^2 \geq 0 \tag{12}$$

holds for some $A > 0$ and $\alpha > 0$, (cf. (4)), and that f is finite. Suppose that there exist bounded derivatives of order k ($k \geq 2$) of all coefficients of (1) and of f. Then there exists a unique solution $u(t, x)$ of the Cauchy problem (1), (2) in the class $W^k_2(G)$, satisfying (1) almost everywhere in G and the conditions (2) in the mean. If $2(k-2) > m+1$, then there exists a classical solution of problem (1), (2).

Proof. The existence of such a solution follows from Lemma 1, if for the coefficients (3) one takes the averages of the corresponding coefficients (1) with averaging radius ε. Condition (5) follows from (12) and the estimate for the square of the averaged function. The uniqueness of the solution of problem (1), (2) is proved in exactly the same way as estimate (9) was obtained.

2. Boundary-value problem. Consider the boundary-value problem for equation (1) in the cylinder $Q\{0 \leq t \leq T,\ x \subset \Omega\}$, where Ω is a domain in R_m with boundary σ, under conditions (2) and the boundary condition

$$u|_S = 0, \qquad S = \{[0,T] \times \sigma\}. \tag{13}$$

By a **generalized solution of the boundary-value problem** (1), (2), (13) in Q we shall mean a function u from $\mathcal{L}_2(Q)$ satisfying the integral identity

$$[L^*(v), u]_Q = [v, f]_Q, \tag{14}$$

where v is an arbitrary function from $\overset{\circ}{W}{}_2^2(Q)$ with the conditions

$$v|_{t=T} = 0, \qquad v_t|_{t=T} = 0, \qquad v|_S = 0. \tag{15}$$

Here

$$L^*(v) \equiv -v_{tt} + (a^{ij} v_{x_i})_{x_j} - (b^{iv})_{x_i} - (b^0 v)_t + cv.$$

Let $\sigma \subset A^{(3)}$.

Theorem 2. Suppose the coefficients of equation (1) and of the equation $L^*(u) = 0$, as well as f and a_t^{ij}, are bounded in Q, and condition (12) is fulfilled. Then there exists a generalized solution of the boundary-value problem (1), (2), (13).

The proof is easily obtained by using the estimate for solutions of problem (3), (2), (13) analogous to (9).

We note that from this estimate it is easy to obtain that, for the generalized solution of (1), (2), (13), the inequality

$$[u, u]_{Q_\delta} \leq M_0 \delta^4, \tag{16}$$

holds, if $[f, f]_{Q_\delta} \leq M_{10} \delta^3$. Here $Q_\delta = \{0 \leq t \leq \delta,\ x \subset \Omega\}$.

Theorem 3. Suppose that in Q the coefficients (1) belong to C^3, that (12) is fulfilled, and

$$|a_t^{ij} \xi_i \xi_j| + |a_{tt}^{ij} \xi_i \xi_j| + |b_t^i \xi_i|^2 \leq K_1 a^{ij} \xi_i \xi_j, \tag{17}$$

and suppose that in some neighborhood of S

$$(a^{ij}_{x_\rho}\xi_i)^2 + (a^i\xi_i)^2 \le K_2 a^{ij}\xi_i\xi_j, \tag{18}$$

where $K_i > 0$ are certain constants. Then the generalized solution of problem (1), (2), (13) is unique. If conditions (17), (18) are fulfilled in $Q^\tau(\tau \le t \le T,\ x \subset \Omega)$ with K_i depending on τ, then the solution of problem (1), (2), (13) is unique in the class of functions satisfying (16).

Proof. Consider in Q a solution of the equation

$$L^*(v) + \varepsilon\Delta v = \Phi \tag{19}$$

with conditions (15), where $\Phi \subset C_0^\infty$ in Q. Suppose (17) is fulfilled. Multiply (19) by $e^{\theta_1 t}$, integrate over Q^τ, and transform the individual terms by integration by parts. For sufficiently large $\theta_1 > 0$ we obtain, taking (12) into account, that

$$[v,v]_{Q^\tau} + [v_t,v_t]_{Q^\tau} + \varepsilon[v_{x_i},v_{x_i}]_{Q^\tau} + (a^{ij}v_{x_i},v_{x_j})_{t=\tau} + [a^{ij}v_{x_i},v_{x_j}]_{Q^\tau} \le M_{11}. \tag{20}$$

To estimate $[v_{tt},v_{tt}]_{Q^\tau}$, we differentiate (19) with respect to t, multiply by $e^{\beta_1 t}v_{tt}$, and integrate it over Q^τ. We shall now estimate $[\varepsilon\Delta v,\varepsilon\Delta v]_Q$. To this end we multiply (19) by $\varepsilon\Delta v$ and integrate over Q. We have

$$(1-3\delta)[\varepsilon\Delta v,\varepsilon\Delta v]_Q \le \delta^{-1}[v_{tt},v_{tt}]_Q + \delta^{-1}[(b^0v)_t - cv + \Phi,(b^0v)_t - cv + \Phi]_Q + \delta^{-1}[(b^{iv})_{x_i},(b^{iv})_{x_i}]_Q - [\varepsilon\Delta v,(a^{ij}v_{x_i})_{x_j}] \tag{21}$$

The penultimate integral in (21) is bounded uniformly with respect to ε by virtue of (12) and (20). Next we have

$$[\varepsilon\Delta v,\varepsilon\Delta v]_Q = \varepsilon[a^{ij}v_{x_kx_i},v_{x_kx_j}]_Q - \frac{1}{2}[\varepsilon a^{ij}_{x_kx_k}v_{x_i},v_{x_j}]_Q +$$

$$+\int_S \varepsilon a^{ij}v_{x_k}\{v_{x_ix_j}\cos(n,x_k) - v_{x_ix_k}\cos(n,x_j)\}\,dS +$$

$$+\int_S \{(\varepsilon v_{x_k}a^{ij}_{x_j}v_{x_i} + \frac{1}{2}\varepsilon a^{ij}_{x_k}v_{x_i}v_{x_j})\cos(n,x_k) - \varepsilon a^{ij}_{x_k}v_{x_k}v_{x_i}\cos(n,x_j)\}\,dS. \tag{22}$$

The first integral on the right in (22) is nonnegative, and the second is bounded by virtue of (20). To estimate the integrals over S, introduce on S a partition of unity: $1 = \sum_l \psi_l$, such that in a neighborhood S_l, where $\psi_l \ne 0$, one can introduce local coordinates $t, y_1, \dots, y_m$ such that S_l lies in the plane $y_m = 0$. Passing to $y_1, \dots, y_{m-1}$ in the last integral I_1 in (22), we obtain

$$I_1 = \sum_l \int_{S_l} \varepsilon \psi_l \chi_\rho^{ij} a_{x_\rho}^{ij} v_{y_m}^2 \, dt \, dy' = -\sum_l \int_Q \varepsilon \frac{\partial}{\partial y_m} (\psi_l \varphi_\gamma \chi_\rho^{ij} a_{x_\rho}^{ij} v_{y_m}^2) \, dt \, dy =$$

$$= \sum_l \left\{ -\int_Q 2\varepsilon \psi_l \varphi_\gamma \chi_\rho^{ij} a_{x_\rho}^{ij} v_{y_m} v_{y_m y_m} \, dt \, dy + \int_Q \varepsilon B_l(t,y) v_{y_m}^2 \, dt \, dy \right\}. \qquad (23)$$

Here $dy' = dy_1 \ldots dy_{m-1}$; $\varphi_\gamma(y_m)$ is a smooth function; $\varphi_\gamma = 1$ for $y_m \leq \gamma$ and $\varphi_\gamma = 0$ for $y_m \geq 2\gamma$; χ_ρ^{ij}, B_l are bounded. The last integral in (23) is bounded uniformly with respect to ε by virtue of (20). To estimate the penultimate integral in (23) we use (8), (18), (20), and the known inequality $\varepsilon^2 [v_{x_i x_j}, v_{x_i x_j}]_Q \leq M_{12} [\varepsilon \Delta v, \varepsilon \Delta v]_Q$, see (12). We write the first integral I_0 over S on the right-hand side of (22) in local coordinates, transform the terms containing $v_{y_m y_\rho}$ $(\rho \neq m)$ by integration by parts, and then estimate I_0 in the same way as I_1. Finally, from (21) we have $[\varepsilon \Delta v, \varepsilon \Delta v]_Q \leq M_{13}$. Substitute the solution v of problem (19), (15) into (14) for $f \equiv 0$. We obtain $[\Phi, u]_Q = [\varepsilon \Delta v, u]_Q$. The last integral tends to zero as $\varepsilon \to 0$ (see (8)). Consequently, $u \equiv 0$. If, in conditions (17) and (18), the constants K_i depend on τ, then $[\varepsilon \Delta v, \varepsilon \Delta v]_{Q^\tau} \leq \widetilde{M}(\tau)$, and in (14), for $f \equiv 0$, we substitute the function $v = \bar{v} \eta_\delta(t)$, where $\bar{v}$ is the solution of problem (19), (15), while $\eta_\delta(t) = 0$ for $t \leq \delta$, $\eta_\delta = 1$ for $t \geq 2\delta$, $0 \leq \eta_\delta \leq 1$, $|\eta_\delta''| = O(\delta^{-1})$, $|\eta_\delta'| = O(\delta^{-2})$. In doing so one must take into account that $[v, v]_{Q^\delta} + [v_t, v_t]_{Q^\delta} = O(\delta)$, as well as (16).

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Note: Figure translations are in progress. See original paper for figures.