ON A PROBLEM WITH AN OBLIQUE DERIVATIVE

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.946.2

MATHEMATICS

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ON A PROBLEM WITH AN OBLIQUE DERIVATIVE

(Presented by Academician I. G. Petrovskii on 6 I 1966)

In a bounded closed domain G with smooth boundary Γ in n-dimensional space, we consider the elliptic equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \tag{1}$$

with the boundary condition

$$\left.\frac{\partial u}{\partial \mu}\right|_{\Gamma} = \left.\sum_{j=1}^{n} \mu_j(x)\frac{\partial u}{\partial x_j}\right|_{\Gamma} = g(x). \tag{2}$$

The coefficients of the equation and the functions $\mu_j(x)$ are assumed, for simplicity, to be infinitely differentiable.

In the case where the direction of the field μ is nowhere tangent to the boundary, the problem under consideration has been well studied. We shall allow the field μ to be tangent to the boundary Γ at points of certain smooth manifolds Γ_i ($i = 0, 1, \ldots, k$), having dimension $n-2$. Let P be an arbitrary point of the manifold Γ_i. By means of a transformation of the independent variables one can ensure that this point is the origin of coordinates and that, in some neighborhood of the point P, the domain G is defined by the inequality $x_n \geq 0$, while the manifold Γ_i is given by the equation $x_n = x_{n-1} = 0$. In the new coordinates condition (2) has the form

$$\left.\sum_{j=1}^{n} \nu_j(x)\frac{\partial u}{\partial x_j}\right|_{\Gamma} = g(x). \tag{2'}$$

In the work (1) L. Hörmander showed that in the case where the principal part of the operator L is the Laplace operator and

$$\sum_1^n \nu_j \frac{\partial \nu_n}{\partial x_j} \leq r\nu_n$$

with some smooth function r, there exists a solution $u \in H_s(G)$ of problem (1)–(2), if $f \in H_{s-1}(G)$ and $g \in H_{s-1/2}(\Gamma)$. If, however,

$$\sum_{j=1}^n \nu_j(x) \frac{\partial \nu_n}{\partial x_j} \geq r\nu_n,$$

then every solution u of such a problem which is equal to zero outside a sufficiently small neighborhood of a boundary point belongs to the space $H_s(G)$. Moreover, he showed that if at points where $\nu_n = 0$,

$$\sum_1^n \nu_j \frac{\partial \nu_n}{\partial x_j} < 0,$$

then problem (1)–(2) has an infinite-dimensional kernel, while if

$$\sum_1^n \nu_j \frac{\partial \nu_n}{\partial x_j} > 0,$$

then it has an infinite-dimensional cokernel.

Some special cases of problem (1)–(2) were considered in the works of A. V. Bitsadze ($^{2-4}$) and Janusauskas (^{5}).

1. We shall assume that the field μ at the points of the manifolds Γ_i is not tangent to these manifolds. In addition, for simplicity we shall assume that there is only one manifold Γ_0 on which the field is tangent to the boundary Γ. In view of our conditions, $v_{n-1}(x) \neq 0$ on Γ_0. We may therefore assume that $v_{n-1}(x) > 0$. The properties of the solutions of problem (1)–(2) depend essentially on the sign of the function $v(Q) = x_{n-1} v_n$ in a neighborhood of the point P.
2. **Theorem 1.** *If everywhere in a neighborhood of P the function $v(Q) \leq 0$, then:*

a) *for $u(x) \in H_s(G)$ the estimate*

$$\|u\|_s^G \leq C\left(\|f\|_{s-1}^G + \|g\|_{s-1/2}^\Gamma + \|u\|_0^G\right) \qquad (s > 1/2); \tag{3}$$

b) *the space of solutions of the homogeneous problem (1)–(2) belonging to $H_s(G)$ has finite dimension;*

c) *the intersection of the range of the operator*

$$(L, \partial/\partial v): \quad H_s(G) \to H_{s-2}(G) \times H_{s-3/2}(\Gamma)$$

with the space $H_{s-1}(G) \times H_{s-1/2}(\Gamma)$ *is closed;*

d) *if* $f \in H_s(G)$, $g \in H_{s+1/2}(\Gamma)$, *then every solution* u *of problem (1)–(2) from* $H_s(G)$ *belongs to the space* $H_{s+1}(G)$.

The following example shows the impossibility, in the case under consideration, of obtaining estimates for the solution of problem (1)–(2) with the left-hand side of inequality (3) replaced by $\|u\|^G_{s+\delta}$ for $\delta > 0$.

Let the boundary of a three-dimensional domain G in a neighborhood of the origin coincide with the plane $x_3 = 0$. Consider the sequence of functions

$$u_m(x_1, x_2, x_3) = e^{(ix_2 - x_3)m} \varphi\left(rm^{1/(p+1)}\right),$$

where the function

$$\varphi(r) = \varphi\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) \in C^\infty(G)$$

vanishes outside this neighborhood. It is easy to verify that, as $m \to +\infty$,

$$\|u_m\|^G_s \geq C m^{s-1/2-1/(p+1)},$$

$$\|\Delta u_m\|^G_{s-1} = O\left(m^{s-1/2}\right),$$

$$\left\|x_1^p\, \partial u_m/\partial x_3 + \partial u_m/\partial x_1\right\|^\Gamma_{s-1/2} = O\left(m^{s-1/2}\right).$$

3. Let us now consider the case when $v(Q) \geq 0$ in a neighborhood of Γ_0. In this case the homogeneous problem (1)–(2) has an infinite-dimensional space of solutions. Therefore we shall require that the solution of problem (1)–(2) satisfy the additional condition

$$u|_{\Gamma_0} = u_0(x). \tag{4}$$

Theorem 2. *If* $v(Q) \geq 0$ *in a neighborhood of* Γ_0, *then:*

a) *for* $u(x) \in H_s(G)$ *the estimate*

$$\|u\|^G_s \leq C\left(\|f\|^G_{s-1} + \|g\|^\Gamma_{s-1/2} + \|u_0\|^\Gamma_{s-1/2} + \|u\|^G_0\right) \tag{5}$$

holds for all $s > 1$;

b) *the space of solutions of the homogeneous problem (1), (2), (4) belonging to* $H_s(G)$, $s > 1$, *is finite-dimensional;*

c) *the intersection of the range of the operator*

$$u \mapsto (Lu, \partial u/\partial v|_\Gamma, u|_{\Gamma_0}),$$

acting from $H_s(G)$ to

$$H_{s-2}(G) \times H_{s-3/2}(\Gamma) \times H_{s-1}(\Gamma_0),$$

with the space

$$H_{s-1}(G) \times H_{s-1/2}(\Gamma) \times H_{s-1/2}(\Gamma_0)$$

is closed;

d) *if $f \in H_s(G)$, $g \in H_{s+1/2}(\Gamma)$, $u_0 \in H_{s+1/2}(\Gamma_0)$, then every solution of problem (1), (2), (4) from $H_s(G)$ belongs to the space $H_{s+1}(G)$;*

e) *if $f \in H_{s-1}(G)$, $g \in H_{s-1/2}(\Gamma)$, $u_0 \in H_{s-1/2}(\Gamma_0)$, and the vector (f, g, u_0) is orthogonal to some finite-dimensional subspace in*

$$H_{s-1}(G) \times H_{s-1/2}(\Gamma) \times H_{s-1/2}(\Gamma_0),$$

then there exists a solution u of problem (1), (2), (4) belonging to the space $H_s(G)$.

As follows from the next example, in the left-hand side of inequality (5) one cannot replace $\|u\|_s^G$ by $\|u\|_{s+\delta}^G$ for $\delta > 0$.

Example. In the domain $G = \{(x_1, x_2, x_3): \; x_1^2 + x_2^2 + x_3^{2p} \leqslant 1\}$ $(p \geqslant 1)$ the sequence of harmonic functions

$$u_m = (x_1 + ix_2)^m x_3$$

satisfies, as $m \to \infty$, the inequalities

$$\|u\|_s^G \geqslant cm^{s-1/2-3/4p},$$

$$\|\partial u/\partial x_3\|_{s-1/2}^\Gamma = O(m^{s-1/2-1/4p}),$$

$$\|u_0\|_{s-1/2}^\Gamma = 0.$$

4. Let now $\nu(Q)$ change sign as Q passes through Γ_0 (i.e., the function $\nu_n(x)$ has a constant sign in a neighborhood of Γ_0).

Theorem 3. If $\nu(Q)$ changes sign when passing through Γ_0, then:

a) for functions $u(x) \in H_s(G)$ the estimate

$$\|u\|_s^G \leqslant C\left(\|f\|_{s-1}^G + \|g\|_{s-1/2}^\Gamma + \|u\|_0^G\right) \qquad (s > 1/2); \tag{6}$$

holds;

b) the space of solutions of the homogeneous problem (1)–(2) belonging to $H_s(G)$ is finite-dimensional;

c) the intersection of the range of the operator $(L, \partial/\partial\nu) : H_s(G) \to H_{s-2}(G) \times H_{s-3/2}(\Gamma)$ with the space $H_{s-1}(G) \times H_{s-1/2}(\Gamma)$ is closed;

d) if $f \in H_{s-1}(G)$, $g \in H_{s-1/2}(\Gamma)$, then every solution u of problem (1)–(2) from $\dot{H}_s(G)$ belongs to the space $H_{s+1}(G)$;

e) if $f \in H_{s-1}(G)$, $g \in H_{s-1/2}(\Gamma)$, and the vector (f, g) is orthogonal to a certain finite-dimensional subspace in $H_{s-1}(G) \times H_{s-1/2}(\Gamma)$, then there exists a solution u of problem (1)–(2) belonging to the space $H_s(G)$.

Estimate (6) cannot be improved. This follows from the example in item 3, if the parameter p takes even values.

5. The main point of the proof consists in studying the solution in a sufficiently small neighborhood of a point P belonging to Γ_0. In this neighborhood we consider the function $w = \partial u/\partial\mu^*$, where the field μ^* is a local extension of the field μ given on the boundary. The function w satisfies a certain linear equation, "close" to equation (1), and boundary conditions on Γ of Dirichlet type. Using the results of the theory of elliptic boundary-value problems, we obtain the required estimates for the function w, and further, depending on the structure of the field μ in a neighborhood of the point P, obtain the corresponding theorems for the solution of problem (1)–(2).

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Received
3 I 1966

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Note: Figure translations are in progress. See original paper for figures.