ON FINITELY MULTIPLE OPEN MAPPINGS

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ON FINITELY MULTIPLE OPEN MAPPINGS

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Here finitely multiple and k-multiple* mappings of topological spaces** are considered. It has turned out that finitely multiple open mappings constitute an important case of zero-dimensional mappings, naturally singled out into a special class, since assertions valid for finitely multiple mappings are false, for example, for compact*** mappings.

The main result should be considered Theorem 1, which apparently does not hold for compact open mappings; this, however, has remained unknown. It is useful to compare Theorem 1 with S. Mardešić's theorem stating that the weight of a locally connected bicompactum is not decreased under a zero-dimensional mapping, and also with its strengthenings given in ([6]).

Theorem 1. *Under a finitely multiple open mapping the weight of a locally bicompact space is not decreased.**** **

Lemma. *Let there be an open k-multiple mapping f of a space X onto Y. Then the weight of Y is not less than the weight of X.*

We prove the lemma. For every point $y \in Y$ there exists a neighborhood O such that

$$f^{-1}Oy = \bigcup_{i=1}^{k} Ox_i, \qquad Ox_{i_1} \cap Ox_{i_2} = \Lambda, \quad i_1 \neq i_2,$$

where $\bigcup_{i=1}^{k} x_i = f^{-1}y$, and such that $fOx_i = Oy$, $i = 1, \dots, k$. Indeed, if $O'x_i$, $i = 1, \dots, k$, are pairwise disjoint neighborhoods, then one may put

$$Oy = \bigcap_{i=1}^{k} fO'x_i,$$

and then $Ox_i = O'x_i \cap f^{-1}Oy$, $i = 1, \dots, k$. Note that on each Ox_i the mapping f is topological.

Such a neighborhood Oy will be called a marked neighborhood of the point y. The system $u_y = \{O_\alpha y\}$ of all marked neighborhoods of the point y forms a base at this point, as is not hard to verify. The system $u = \{u_y\}$, over all $y \in Y$, forms a base of the space Y.

From the base u choose a base v, whose cardinality is equal to the weight of Y, $v = \{v_\alpha\}$, where every element $v_\alpha \in v$ is a marked neighborhood of some point $y \in Y$. As we have seen,

$$f^{-1}v_\alpha = \bigcup_{i=1}^{k} w_{\alpha i},$$

where each $w_{\alpha i}$ is open in X, and by means of f it is mapped topologically onto all of v_α.

The cardinality of the system $w = \{w_{\alpha i}\}$ over all admissible α and i is obviously equal to the weight of Y. We shall show that all possible pairwise intersections of elements of w form a base of the space X, which will complete the proof of the lemma.

Let Ox be an arbitrary neighborhood of some point $x \in X$. There exists $v_\alpha \in v$ such that $y \in v_\alpha \subseteq fOx$, where $y = fx$. Choose $w_{\alpha i}$ such that $x \in w_{\alpha i}$. Denote

$$Oy = f(w_{\alpha i} \cap Ox);$$

there exists $v_\beta \in v$ such that $x \in v_\beta \subseteq Oy$. Choose $w_{\beta j}$ such that $x \in w_{\beta j}$. Let us verify that

$$x \in w_{\alpha i} \cap w_{\beta j} \subseteq Ox.$$

* A mapping is finitely multiple if the preimage of every point consists of a finite number of points. A mapping is k-multiple if the preimage of every point consists of exactly k points.

** Everywhere in this paper, by a space is meant a Hausdorff space, and by a mapping—a continuous mapping.

*** A mapping is compact if the preimage of every point is compact.

**** Obviously, the weight is not increased either.

If it were the case that $(w_{\alpha i} \setminus Ox) \cap w_{\beta j} \neq \Lambda$, then we would have $v_\beta \cap f(w_{\alpha i} \setminus Ox) \neq \Lambda$, but this is not so. Hence $(w_{\alpha i} \setminus Ox) \cap w_{\beta j} = \Lambda$, whence it follows that $w_{\alpha i} \cap w_{\beta j} \subseteq Ox$. The lemma is proved.

We shall now prove the theorem without difficulty. There is an open finite-to-one mapping $f : X \to Y$, where X is locally bicompact; $Y = \bigcup_{k=1}^{\infty} Y_k$, where Y_k consists only of points of multiplicity k of the mapping f. Correspondingly

$X = \bigcup_{k=1}^{\infty} X_k$, where $X_k = f^{-1}Y_k$; the mapping $f : X_k \to Y_k$ is open and k-to-one. In view of the lemma, the weight of $X_k \leq$ the weight of Y_k. Applying the known sum theorem from ([1]), we obtain that the weight of $X \leq$ the weight of Y.

Theorem $1'$**.** *Let there be a finite-to-one open mapping $f : X \to Y$, where X is an m-compact space, and the weight of $Y \leq m$. Then the weight of $X \leq m$.*

Corollary. *If a bicompactum is mapped finite-to-one and openly onto a compactum, then such a bicompactum is metrizable.*

In connection with these facts there arises a question, the answer to which is unknown: does there exist a nonmetrizable bicompactum which is mapped openly and compactly onto a compactum?

The following proposition gives a partial answer to this question (and even to a stronger one).

Proposition 1. *If a dyadic* bicompactum X is mapped compactly onto a compactum, then the bicompactum X is metrizable.*

Indeed, it is easy to show that the bicompactum X will satisfy the first axiom of countability, and then, by a known theorem ([2]), it is metrizable.

In the proof of Theorem 1 only the fact was used that the sum theorem is valid for a locally bicompact space. Therefore the analogous assertion will be true for any space in which the sum theorem holds. In view of this the following propositions are valid.

Theorem $1''$**.** *Under a finite-to-one open mapping, the weight of a p-space* ([7]) *is not lowered.*

Corollary 1. *Under a finite-to-one open mapping, the weight of a metric space is not lowered.*

Corollary 2. *Under a finite-to-one open mapping, the weight of a space complete in the sense of Čech is not lowered.*

Theorem $1'''$**.** *Under a finite-to-one open mapping, the weight of a space having a perfectly normal bicompact extension is not lowered.*

Next we give facts concerning k-to-one open mappings.

Theorem 2. *Let there be an open k-to-one mapping $f : X \to Y$, where Y is a paracompact space. Then X is also a paracompact space.*

Proof. Consider an arbitrary covering $\omega = \{\omega_\alpha\}$ of the space X; inscribe in the covering $f\omega = \{f\omega_\alpha\}$ of the space Y a special covering $u = \{u_\alpha\}$ as follows. For every point $y \in Y$ there will be found a neighborhood $u_\alpha \ni y$ such that

$$f^{-1}u_\alpha = \bigcup_{i=1}^{k} v_{\alpha i},$$

where each $v_{\alpha i}$ is open and is contained entirely in one of the elements of ω, and, moreover, $fv_{\alpha i} = u_\alpha$, $i = 1, \dots, k$. The covering $u = \{u_\alpha\}$ is inscribed in $f\omega = \{f\omega_\alpha\}$, and the covering $v = \{v_{\alpha i}\}$ is inscribed in $\omega = \{\omega_\alpha\}$.

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* A bicompactum homeomorphic to D^τ (the product of τ copies of the two-point space) is called dyadic. Every dense subspace of a dyadic bicompactum is called a dyadic space.

We inscribe in the cover $u = \{u_\alpha\}$ a locally finite cover $w = \{w_\alpha\}$; for every $w_\alpha \in w$ we choose some one $u_\beta \in u$ such that $w_\alpha \subseteq u_\beta$, and put $\theta_{\alpha i} = f^{-1} w_\alpha \cap v_{\beta i}$, $i = 1, \dots, k$. The cover $\theta = \{\theta_{\alpha i}\}$ is a locally finite cover inscribed in ω.

Corollary. *If a space X is open and k-to-one mapped onto a metrizable space, then it is itself metrizable.*

The space X is locally metrizable, since an open k-to-one mapping is a local homeomorphism. From our Theorem 2 it follows that X is paracompact. But a locally metrizable paracompact space is metrizable ([4]).

Theorem 3. *If a space Y is an open k-to-one image of a paracompact space, then Y itself is paracompact.*

Let $\omega = \{\omega_\alpha\}$ be an arbitrary cover of the space Y; in it, as we have seen, one can inscribe a cover $u = \{u_\alpha\}$ such that

$$f^{-1} u_\alpha = \bigcup_{i=1}^{k} v_{\alpha i}, \qquad f v_{\alpha i} = u_\alpha, \quad i = 1, \dots, k,$$

so that on each $v_{\alpha i}$ the mapping f is topological. In the cover $v = \{v_{\alpha i}\}$ of the space X we inscribe a locally finite $w = \{w_\alpha\}$. It is possible to verify that $fw = \{fw_\alpha\}$ is a locally finite cover inscribed in ω. Indeed, let $y \in Y$ be an arbitrary point of Y; we indicate for it a neighborhood which intersects only finitely many elements of the cover fw;

$$f^{-1} y = \bigcup_{i=1}^{k} x_i.$$

For each point x_i we take a neighborhood Ox_i such that it intersects only finitely many elements of w. The required neighborhood is

$$Oy = \bigcap_{i=1}^{k} f O x_i.$$

The assertion is proved.

Corollary. *If a space Y is an open k-to-one image of a metrizable space, then Y itself is metrizable.*

Theorem $2'$. *Suppose there is an open k-to-one mapping $f : X \to Y$, where Y is a weakly paracompact (strongly paracompact) space. Then X is also weakly paracompact (respectively, strongly paracompact).*

Theorem $3'$. *If a space Y is an open k-to-one image of a weakly paracompact space, then Y itself is weakly paracompact.*

The methods of proof of the last theorems are the same as for Theorems 2 and 3.

Example 1. An example of an open finite-to-one mapping of a nonparacompact space onto a paracompact space (showing that Theorem 2 is false for finite-to-one mappings). As the space Y one takes the space T of all ordinal numbers up to ω_1, inclusive. The space X consists of two isolated copies of the space T, one of which is taken without the point ω_1. The mapping f of the space X onto Y is defined as follows: to each ordinal number $x \in X$ there is assigned the equal ordinal number in Y. The mapping f is open and two-valued (but not 2-to-one!).

Example 2. An example of an open finite-to-one mapping of a nonmetrizable normal space onto a compactum. To the Gol' tsin space ([5]), along the boundary circle, an ordinary disk is glued—thus we obtain the space X. The space X, obviously, two-valuedly and openly "projects" onto the ordinary disk.

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Note: Figure translations are in progress. See original paper for figures.