On Multiply Complete Systems and Non-Self-Adjoint Operators Depending on the Parameter \(\lambda\)

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Abstract

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On Multiply Complete Systems and Non-Self-Adjoint Operators Depending on the Parameter λ

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The concept of multiple completeness of a system of elements was first introduced by M. V. Keldysh in his fundamental studies* concerning non-self-adjoint operators. This concept is given for the system of eigen and associated (e.a.) elements of the operators under study (see ([1])). In the present paper the concept of multiple completeness of a system of elements in a Hilbert space** $\mathcal{H}$ is given. Further, certain general criteria are established under which these systems form a multiple basis. The results obtained are applied to the study of non-self-adjoint operators.

1. By $\mathcal{H}^{(n)}$ we shall denote the direct product of n copies of the Hilbert space $\mathcal{H}$ with scalar product

$$[\tilde{x}, \tilde{y}] = [\{x_1, x_2, \dots, x_n\}, \{y_1, y_2, \dots, y_n\}] = \sum_{i=1}^{n} (x_i, y_i),$$

where (x, y) is the scalar product in the space $\mathcal{H}$.

Definition. We shall say that a system of elements $\{\varphi_i\}$ is n-fold complete in the space $\mathcal{H}$ if there exist systems $\{\psi_i^{(k)}\}$ $(k = 1, 2, \dots, n-1)$ such that $\{\varphi_i, \psi_i^{(1)}, \dots, \psi_i^{(n-1)}\}$ form a complete system in the space $\mathcal{H}^{(n)}$.

It is easy to show that if e_i is a complete orthonormal system, and the matrix $\|\alpha_{i,k}^{(j)}\|$ is invertible for every i, then

$$\Phi_{i,k} = \{\alpha_{i,k}^{(1)} A_k^{(1)} e_i,\ \alpha_{i,k}^{(2)} A_k^{(2)} e_i,\ \dots,\ \alpha_{i,k}^{(n)} A_k^{(n)} e_i\}$$

will form different classes of bases or complete systems in the space $\mathcal{H}^{(n)}$, depending on the properties of the operator $\tilde{A}$ acting in $\mathcal{H}^{(n)}$,

$$\tilde{A} f = \tilde{A}_k f \quad \text{for } f \in \mathcal{H}_k^{(n)},$$

where

$$\tilde{A}_k = \left\| \begin{matrix} A_k^{(1)} & 0 & & \\ & A_k^{(2)} & & \\ & & \ddots & \\ 0 & & & A_k^{(n)} \end{matrix} \right\|, \qquad \mathcal{H}^{(n)} = \sum_{k=1}^{n} \mathcal{H}_k^{(n)},$$

$\mathcal{H}_k^{(n)}$ is the minimal subspace containing the system of elements

$$\Psi_{i,k} = \{\alpha_{i,k}^{(1)} e_i, \alpha_{i,k}^{(2)} e_i, \dots, \alpha_{i,k}^{(n)} e_i\}.$$

Let φ_i form a Riesz basis in the space $\mathcal{H}$, and let i_k be some increasing sequence of natural numbers. Let us compose a new system

$$\psi_{i_k+j} = \sum_{l=0}^{i_{k+1}-i_k} a_{j,l}^{(i_k)} \varphi_{i_k+l}, \qquad 0 \le j \le i_{k+1} - i_k.$$

If the matrix $a_{j,l}^{(i_k)}$ is invertible for every k, then ψ_i forms a Riesz basis with parentheses in $\mathcal{H}$. Co-

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* Some of these results were published in ([1]), but some of the results concerning the study of resolvents of non-self-adjoint operators are absent from that paper, although they were used in obtaining the results of paper ([1]). Partly these results are presented and used also in ([14–16]).

** This concept of multiply complete systems is also suitable for Banach spaces.

the corresponding assertion holds also in the case when φ_i forms a Riesz basis with parentheses.

Using the assertions set forth above, and analogous ones which, for lack of space, we do not present, one can establish many properties of multiply complete systems. As an example we give a theorem which is very useful in the study of non-self-adjoint operators.

Theorem 1. Let $\Phi_i = \{\varphi_i^{(0)}, \varphi_i^{(1)}, \dots, \varphi_i^{(n-1)}\}$ be a complete system in $\mathcal{H}^{(n)}$, which is decomposed into n subsystems in such a way that in each subsystem $a_{ik}^{(j)} \varphi_{i_k}^{(j)}$ $(j = 0, \dots, n-1)$ form a Riesz basis in $\mathcal{H}$, where $a_{ik}^{(j)}$ are some numbers. Then the Φ_i form a basis with parentheses, and, under a suitable normalization, a Riesz basis with parentheses in $\mathcal{H}^{(n)}$.

2. Consider the operator

$$A(\lambda) = \sum_{i=1}^{n} \lambda^i H^i A_i (E - A_0)^{-1} + \sum_{k=1}^{m} \lambda^{-k} T^k B_k (E - B_0)^{-1}, \tag{1}$$

where H and T are completely continuous positive* self-adjoint operators, A_i $(i < n)$, B_j $(j < m)$ are completely continuous operators, $A_n = B_m = E$, and

E is the identity operator. The eigenvalues (e.v.) of the operator $\lambda^n H^n$ lie outside the circle $|\lambda| = 1/\|H\|$ on the rays $\arg\lambda = 2k\pi/n,\ k = 0, 1, \dots, n-1$, while the e.v. of the operator $\lambda^{-m}T^m$ lie inside the circle $|\lambda| = \|T\|$ on the rays $\arg\lambda = -2k\pi/m,\ k = 0, 1, \dots, m-1$.

Let $G(r, k, \varphi, \psi)$ be a certain domain of the complex plane consisting of the points λ: $\{\lambda : |\lambda| \geq r,\ 2k\pi/n - \varphi \leq \arg\lambda \leq 2k\pi/n + \psi\}$; analogously, $M(r, k, \varphi, \psi) = \{\lambda : |\lambda| \leq r,\ -2k\pi/m - \varphi \leq \arg\lambda \leq -2k\pi/m + \psi\}$. We shall say that the system $\{e_i\}$ is almost complete if the subspace of all elements orthogonal to all e_i is finite-dimensional. This subspace will be called the defect of the system $\{e_i\}$.

Theorem 2. Suppose at least one of the following two conditions is satisfied: a)

$$\varliminf_{i\to\infty} i|\mu_i|^p < \infty; \qquad \varliminf_{i\to\infty} i|\lambda_i|^q < \infty;$$

A_{iH}^{-q} $(i = 0, \dots, n-1)$, B_{iT}^{-p} $(i = 0, \dots, m-1)$ are completely continuous; b)

$$\varliminf_{i\to\infty} i|\mu_i|^p = 0; \qquad \varliminf_{i\to\infty} i|\lambda_i|^q = 0;$$

$H^{-q}A_i,\ T^{-p}B_j$ $(j = 0, \dots, m-1;\ i = 0, \dots, n-1)$ are bounded, where μ_i, λ_i are the e.v. of the operators T, H, respectively, and $p > 0,\ q > 0$.

Then the system of eigenvectors of the operator (1) corresponding to the e.v. λ_i lying in each of the domains $G(r, k, \varphi, \psi)$ or $M(r, k, \varphi, \psi)$, for every $0 < r < \infty,\ \varphi > 0,\ \psi > 0$, forms an almost complete system, and for $\varphi < \pi/n,\ \psi < \pi/n,\ R$ sufficiently large for $G(R, k, \varphi, \psi)$ and $\varphi < \pi/m,\ \psi < \pi/m,\ r$ sufficiently small for $M(r, k, \varphi, \psi)$, it is a Riesz basis with parentheses in the closure of its linear span.

Remark. In the case when r is sufficiently large, the system of eigenvectors of the operator $A(\lambda)$ corresponding to the e.v. lying in $G(r, k, \varphi, \psi)$, and the eigenvectors of the operator $\lambda^n H^n$ corresponding to the e.v. $\lambda_i,\ |\lambda_i| \leq r,\ \arg\lambda_i = 2\pi k/n$, together form a Riesz basis with parentheses in $\mathscr{H}$. An analogous assertion is also true for the domain $M(r, k, \varphi, \psi)$ and the operator $\lambda^{-m}T^m$, if r is sufficiently small and $|\lambda_i| \geq r,\ \arg\lambda_i = -2\pi k/m$.

Using Theorem 2 and the results of § 1 on multiply complete systems, one can establish theorems on the convergence of expansions in eigenvectors corresponding to different sets of eigenvalues of the operator. In particular, under the conditions of the theorem the system of eigenvectors of the operator $A(\lambda)$

* Only for convenience of exposition we assume that H and T are positive,

but in fact the analogous assertion is true for any self-adjoint and even some normal operators. Moreover, the assertion of the theorem concerning the domain $G(r, k, \varphi, \psi)$ remains valid for the operator

$$A(\lambda) = B(\lambda) + \sum \lambda^i H^i A_i (E - A_0)^{-1}$$

under the same conditions on A_i and H $(i = 0, \dots, n)$, if one requires that $B(\lambda)$ be analytic in $G(r, k, \varphi, \psi)$ and satisfy the condition $\|B(\lambda)\| \leq M/|\lambda|$. The analogous statement is also valid for $M(r, k, \varphi, \psi)$.

forms an $(m + n)$-fold basis with parentheses, and under the corresponding normalization becomes a Riesz basis with parentheses. The case $n = m = 1$ is of special interest, since it has applications in mechanics (see ($^{5-10}$)).

Let A and B be complete self-adjoint operators, and let λ_i and μ_i be their eigenvalues, respectively. Suppose that the following conditions are satisfied:

a)

$$\sum_{i=1}^{\infty} \frac{|\lambda_i|^4}{\delta_i} < \infty, \qquad \sum_{i=1}^{\infty} \frac{|\mu_i|^4}{\varepsilon_i} < \infty, \quad \text{where} \quad \delta_i = \inf_{j \neq i} |\lambda_i - \lambda_j|, \quad \varepsilon_i = \inf_{j \neq i} |\mu_i - \mu_j|;$$

b) the multiplicity of all eigenvalues of the operators A and B, except for a finite number, is equal to 1.

Theorem 3. *Under conditions a) and b), the system of eigen- and associated elements of the operator $A/\lambda + \lambda B$ forms a double Bari basis; except for a finite number, all eigenvalues are simple.*

Theorem 4. *If the operators A, B are complete self-adjoint operators of finite (generally different) orders, then the system of eigen- and associated elements forms a double Riesz basis with parentheses.*

Remark. If the completeness condition for A and B is dropped, then the system of eigen- and associated elements is a double Riesz basis with parentheses in the closure of its linear span, whose defect dimension coincides with the sum of the defects* of the operators A and B.

Remark. If A and B are completely continuous normal operators of finite orders with eigenvalues on a finite number of rays ξ_i $(i = 1, \dots, m)$, η_i $(i = 1, \dots, l)$, respectively, then Theorem 4 and Remark 1 remain valid; moreover, the eigenvalues of the operator $A/\lambda + \lambda B$ tend asymptotically to the rays ξ_i in a neighborhood of the origin and to the rays η_i in a neighborhood of the point at infinity.

If the operator A has only a finite number of positive eigenvalues (negative eigenvalues), then we shall call the operator A almost negative (almost positive). If A and B are completely continuous self-adjoint operators, then the operator $A/\lambda + \lambda B$ can have only a finite number of complex eigenvalues.

Theorem 5. *Let the operators A and B have finite orders. In order that the real parts of all eigenvalues of the operator $A/\lambda+\lambda B$ be positive (negative), it is necessary that the operators A and B be almost positive (almost negative), and it is sufficient that the operators A and B be positive (negative).*

As an application we consider the equation**

$$\frac{d}{dz}\left\{\left[\rho(z)-\frac{\mu(z)}{n}\left(\frac{d^2}{dz^2}-k^2\right)\right]\frac{dW}{dz}-\frac{1}{n}\mu'\left(\frac{d^2}{dz^2}+k^2\right)W\right\}$$
$$=k^2\left\{-\frac{g}{n^2}\rho'W+\left[\rho-\frac{\mu(z)}{n}\left(\frac{d^2}{dz^2}-k^2\right)\right]W-\frac{2}{n}\mu'\frac{dW}{dz}\right\}; \qquad (2)$$

with

$$a\le z\le b, \qquad W(a)=W(b)=W'(a)=W'(b)=0, \qquad (3)$$

where $\rho(z)$ and $\mu(z)$ are known functions of z; k is a constant; $1/n=\lambda$ is a parameter, g is a constant.

We rewrite equation (1) in a form convenient for investigation:

$$L(k)W=\frac{1}{k^2}\frac{d^2}{dz^2}\left(\mu\frac{d^2W}{dz^2}\right)-2\frac{d}{dz}\left(\mu\frac{dW}{dz}\right)+k^2\mu W$$
$$=\frac{1}{\lambda}\left(\frac{1}{k}\frac{d}{dz}\left(\rho\frac{dW}{dz}\right)-\rho W\right)+\lambda g\rho' W\equiv\frac{1}{\lambda}M(k)W+\lambda CW. \qquad (4)$$

* The defect of the operator A is the dimension of the subspace $H(A)$ for which $Af=0$ when $f\in H(A)$.

** This equation is obtained in the study of fluid motion; see ([5,9,10]).

Suppose that $k^2\mu+\mu''>0$ almost everywhere; then $L(k)$, under conditions (3), is a positive definite self-adjoint operator, $L^{-1}(k)$ is completely continuous and its order does not exceed 1/4. It is easy to show that $L^{-1/2}ML^{-1/2}$, where $L^{-1/2}$ is the positive root of L^{-1}, is a self-adjoint operator of order not higher than 1/2, and the sign of the operator $L^{-1/2}ML^{-1/2}$ is determined by the sign of the function ρ. $L^{-1/2}CL^{-1/2}$ is a self-adjoint operator of order 1/4, and its sign is determined by the sign of the function ρ'. Thus, in the case $k^2\mu+\mu''>0$, problem (2), (3) is equivalent to the equation

$$y=\frac{1}{\lambda}L^{-1/2}ML^{-1/2}y+\lambda L^{-1/2}CL^{-1/2}y \qquad (5)$$

in the space $\mathcal{L}_2(a,b)$.

Theorem 6. *For all real $k\neq 0$, equation (5) has a twofold complete system of eigenfunctions, forming a twofold basis of Bari.*

The eigenvalues tend to 0 *and to* ∞, *and only a finite number of eigenvalues can be complex; all real eigenvalues and the real parts of all eigenvalues are negative if* $\rho > 0$, $\rho' < 0$. *If* $\rho < 0$ *on a set of positive measure or* $\rho' > 0$ *on a set of positive measure, then equation (5) has an infinite set of positive eigenvalues.*

This theorem completely solves the problem of stability of the motion of a fluid posed in ([5]). For lack of space we do not present the formulation of the problem and the result obtained in terms of mechanics.

In the case where the operator $A(\lambda)$ depends polynomially on λ, under conditions equivalent to those of Theorem 2, the convergence of n-fold expansions in eigenfunction elements for elements from a certain everywhere dense set was proved by R. M. Dzhabar-zade ([11]), and subsequently independently by V. N. Vizitei and A. S. Markus ([13]). A stronger result is contained in ([3]). In the case of the operator $A/\lambda + \lambda B$, some results on the convergence of twofold expansions under very stringent restrictions on A and B are given in ([7]), see also ([6]). In ([9]) (see also ([10])) an operator of the form* $A(\lambda)/\lambda + \lambda B$ is considered, where $A(\lambda)$ is uniformly bounded for sufficiently large $|\lambda|$, and, under some other conditions, it is shown that there is an infinite number of eigenvalues and that the corresponding system of eigenfunction elements is almost complete and quadratically close to a certain orthonormal system.

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* Article ([9]) begins with a reference to problem (2), (3), which the author reduces to the equation $(A/\lambda + \lambda B)y = y$ (a), where A is non-self-adjoint. The consideration of $(A(\lambda)/\lambda + \lambda B)y$ (b) can be explained only by the fact that the method used in ([9]) gives equally incomplete results both for (a) and for (b), insufficient for the investigation of problem (2), (3). In ([9]) there is also an inaccurate assertion about the limit points of the eigenvalues of equation (b). In fact the only possible limit points of eigenvalues can be ∞ and 0. The assertion that the limit points of the eigenvalues are infinity applies to the eigenvalues that are found in ([9]).