HYPOELLIPTIC EQUATIONS IN CONVOLUTIONS

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L. R. VOLEVICH

HYPOELLIPTIC EQUATIONS IN CONVOLUTIONS

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1. In recent years a number of profound works have appeared on the theory of singular integro-differential (pseudodifferential) operators ([1–8]). It has been found that these operators possess many important properties inherent in differential operators; in particular, the "principle of locality" holds for them. This principle makes it possible to construct for pseudodifferential operators not only a theory of boundary-value problems, but also a local theory.

In the present paper the local properties of pseudodifferential operators are studied. We single out a broad class of operators with nonhomogeneous symbol, which is a natural generalization of differential operators of constant strength ([9]). By the methods of ([8]), for this class of operators a calculus is constructed, analogous to the calculus of homogeneous operators. Next, hypoelliptic operators are introduced, generalizing formally hypoelliptic differential operators ([9,10]). An essential role throughout the exposition is played by the spaces H^μ of tempered distributions; the theory of these spaces was developed in ([9,11]).

2. Definitions. $x = (x^1, \dots, x^n)$ are points of the n-dimensional real space R^n, $\xi = (\xi_1, \dots, \xi_n)$ are variables dual with respect to the form $x \cdot \xi = x^1\xi_1 + \dots + x^n\xi_n$; $D_k = -i\,\partial/\partial x^k$, $D = (D_1, \dots, D_n)$; $\partial_k = \partial/\partial\xi_k$, $\partial = (\partial_1, \dots, \partial_n)$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a set of n integers (a multi-index), then $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. The quantities D^α, x^α, ∂^α are defined similarly. C^∞, $C^\infty(\Omega)$ are the spaces of infinitely differentiable functions in R^n (in the domain Ω); C_0^∞ ($C_0^\infty(\Omega)$) are the functions from C^∞ ($C^\infty(\Omega)$) with compact supports. $\mathfrak{S}$ is the space of functions $u(x) \in C^\infty$ for which $\sup |x^\alpha D^\beta u(x)| < \infty$ for all multi-indices α, β; $\mathfrak{S}'$ is the conjugate space of tempered distributions. If $u \in \mathfrak{S}'$, then by $\tilde{u}$ we shall denote the Fourier transform of the distribution u, and by $H^{(s)}$ the set of those $u \in \mathfrak{S}'$ for which $\tilde{u}$ is a locally integrable function and $(1 + |\xi|)^s \tilde{u}(\xi) \in L_2$. By $H^{(\infty)}$, $H^{(-\infty)}$ we denote the intersection (union) of all spaces $H^{(s)}$, $-\infty < s < \infty$, endowed with the natural topology of the projective (inductive) limit.

3. Definition of pseudodifferential operators. Pseudolocality. We shall call symbols the functions $a(x, \xi)$, defined for all pairs $(x, \xi) \in R^n \times R_n$ and having a limit as $|x| \to \infty$, i.e. $a(x,\xi) = a(\xi) + a'(x,\xi)$, where $a'(x,\xi) \to 0$ as $|x| \to \infty$. To each symbol, following ([8]), one can associate operators

$$\mathcal{A}u(x) = (2\pi)^{-n/2} \int e^{ix\cdot\eta} a(x,\eta)\tilde{u}(\eta)\, d\eta, \tag{1}$$

$$Au(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \left[(2\pi)^{-n/2} \int e^{-iy\cdot\xi} a(y,\xi) u(y)\, dy \right] d\xi. \tag{2}$$

The operators (1), (2) are initially defined in $\mathfrak{S}$ or $H^{(\infty)}$. Using the formal adjointness of the operators $\mathcal{A}$ and $\bar{A}$ ($\bar{A}$ is the operator

(2)′, corresponding to the symbol $\bar{a}(x,\xi)$, i.e.

$$(\mathcal{A}u, v) = (u, \bar{A}v); \quad u, v \in \mathcal{S}, \tag{3}$$

they can be extended to the dual spaces $\mathcal{S}'$, $H^{(-\infty)}$.

By the methods of ([8]) one proves

Proposition 1. If $a(x,\xi) \in C^\infty$ with respect to x,

$$|a(\xi)| < c(1 + |\xi|)^N, \tag{4}$$

and for all multiindices α

$$\int |D^\alpha a'(x,\xi)|\, dx \le c_\alpha (1 + |\xi|)^{N+\theta|\alpha|}, \quad \theta < 1, \tag{4'}$$

then the operators (1), (2) are continuous as operators acting from $H^{(\infty)}$ to $H^{(\infty)}$ and from $H^{(-\infty)}$ to $H^{(-\infty)}$. If, moreover, $a(x,\xi) \in C^\infty$ with respect to x and ξ, and the derivatives $D^\beta a(x,\xi)$ satisfy conditions (4), (4′) (the numbers N and θ may depend on β), then the operators (1), (2) are continuous as operators acting from $\mathcal{S}$ to $\mathcal{S}$ and from $\mathcal{S}'$ to $\mathcal{S}'$.

An operator $\mathfrak{A} : \mathcal{S}' \to \mathcal{S}'$ is called **pseudolocal** if $\{u \in \mathcal{S}', u = 0 \text{ in } \Omega \Rightarrow \mathfrak{A}u \in C^\infty(\Omega)\}$, where Ω is some domain in R^n. As noted in ([8]), if the operator $\mathfrak{A}$ is pseudolocal and $\mathfrak{A}\mathcal{S} \subset \mathcal{S}$, then $\{u \in \mathcal{S}', u \in C^\infty(\Omega) \Rightarrow \mathfrak{A}u \in C^\infty(\Omega)\}$. Modifying the arguments of ([3]), one can prove

Proposition 2. Let the continuity conditions for the operators (1), (2) in $\mathcal{S}'$ (Proposition 1) be fulfilled, and suppose that for any $s > 0$ there exists $N = N(s)$ such that for all β

$$\sum_{|\alpha|=N} \left[\delta^\alpha a(\xi) + \int |D^\beta \delta^\alpha a'(x,\xi)|\, dx \right] \le c_\beta (1 + |\xi|)^{-s}.$$

Then the operators (1), (2) are pseudolocal.

4. Pseudodifferential operators in the spaces H^μ. We shall denote all functions of the form const $\cdot(1+|\xi|)^N$ occurring below by $\rho(\xi)$. As in ([11]), let $\mathfrak{B}$ denote the class of nonnegative weight functions satisfying the conditions: for all $\xi, \eta \in R_n$,

$$\mu(\xi)\mu^{-1}(\eta) \leq \rho(\xi-\eta). \tag{5}$$

By H^μ we denote the set of such functions $u \in H^{(-\infty)}$ that $\mu(\xi)\tilde{u}(\xi) \in L_2$. In view of (5), H^μ will be a module over $\mathcal{S}$. By $H^\mu_{\text{loc}}(\Omega)$ we denote the set of such distributions $u \in \mathcal{S}'$ for which $\varphi u \in H^\mu$ for all $\varphi \in C_0^\infty(\Omega)$.

Put

$$\mu_{(s)}(\xi) = (1+|\xi|)^s \mu(\xi). \tag{6}$$

We shall say that μ is the order of the operator $\mathfrak{A} : H^{(\infty)} \to H^{(\infty)}$ equal to r, if $\|\mathfrak{A}u\|_{\nu_{(s)}} \leq C\|u\|_{\mu\nu_{(s+r)}}$ for arbitrary $\nu \in \mathfrak{B}$ and $-\infty < s < \infty$. If $r = 0$ ($r < 0$), then the operator $\mathfrak{A}$ is called μ-bounded (μ-smoothing). If $\mu \equiv$ const, then we shall simply call the operator bounded (smoothing).

Proposition 3. Let $\mu(\xi) \in \mathfrak{B}$ and for all α

$$|a(\xi)| < c\mu(\xi), \qquad \int |D^\alpha a'(x,\xi)|\,dx < c_\alpha \mu(\xi), \tag{7}$$

Then the operators (1), (2) are μ-bounded.

5. Regular pseudodifferential operators. Let $\mathfrak{B}_0$ denote the class of weights $\mu \in \mathfrak{B}$ satisfying the additional condition

$$|\mu(\xi) - \mu(\eta)| < \rho(\xi-\eta)(1+|\eta|)^{-\sigma}\mu(\eta), \quad \sigma > 0, \tag{8}$$

and by $\mathfrak{S}_0^\mu$ the class of symbols $a(x,\xi)$ for which inequalities (7) are satisfied and, in addition,

$$|a(\xi) - a(\eta)| < \rho(\xi-\eta)(1+|\eta|)^{-\sigma}\mu(\eta), \tag{9}$$

$$\int |D_a^\alpha a'(x,\xi) - D_a^\alpha a'(x,\eta)|\,dx < c_\alpha \rho(\xi-\eta)(1+|\eta|)^{-\sigma}\mu(\eta). \tag{9'}$$

Symbols $a(x,\xi) \in \mathfrak{S}_0^\mu$ will be called μ-regular.

Proposition 4. If $a(x,\xi) \in \mathfrak{S}_0^\mu$, then $\mathcal{A} - A$ is a μ-smoothing operator.

Proposition 5. If $a(x,\xi) \in \mathfrak{S}_0^\mu$, $b(x,\xi) \in \mathfrak{S}_0^\nu$, then

$$c(x,\xi) = a(x,\xi)b(x,\xi) \in \mathfrak{S}_0^{\mu\nu},$$

and the operator $\mathcal{AB} - \mathcal{C}$ is μ-smoothing. (Here $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the operators (1) corresponding to the symbols a, b, c.)

Corollary. If $a(x,\xi) \in \mathfrak{S}_0^\mu$ and $\varphi(x) \in \mathcal{S}$, then the operator $[\mathcal{A},\varphi]$ is μ-smoothing.

For μ-regular symbols one can refine Proposition 3.

Proposition 6. Let $a(x,\xi) \in \mathfrak{S}_0^\mu$, and suppose that $|a(x,\xi)| \le \mu(\xi)$. Then for every $\varepsilon > 0$ and arbitrary $\nu \in \mathfrak{B}$, $-\infty < s < \infty$,

$$\|\mathcal{A}u\|_{\nu(s)} \le (1+\varepsilon)\|u\|_{\mu\nu(s)} + C(\mu,\nu,s,\varepsilon)\|u\|_{\mu\nu(s-1)}. \tag{10}$$

The proofs of Propositions 3-6 require no new ideas compared with (8).

6. Hypoelliptic pseudodifferential operators

Theorem 1 (on global regularity). Let $a(x,\xi) \in \mathfrak{S}_0^\mu$, and suppose that for sufficiently large ξ the lower bound

$$|a(x,\xi)| > c\mu(\xi), \quad |\xi| > R \tag{11}$$

holds. Then

$$\{u \in H^{(-\infty)},\ \mathcal{A}u \in H^{\nu(s)} \Rightarrow u \in H^{\mu\nu(s)}\}.$$

Proof. Condition (11) makes it possible to construct for the operator (1) a left regularizer, i.e., a $1/\mu$-bounded operator $\mathfrak{R}$ such that

$$\mathfrak{R}\mathcal{A} = E + T,$$

where T is a smoothing operator (for some $\sigma > 0$). As $\mathfrak{R}$ one may take the operator (1) corresponding to the symbol

$$\varphi(\xi)/a(x,\xi),$$

where $\varphi(\xi) \in C^\infty$, $\varphi(\xi) = 0$ for $|\xi| \le R$ and $\varphi(\xi) = 1$ for $|\xi| \ge R+1$. From the inclusion $u \in H^{(-\infty)}$ it follows that $u \in H^{\mu\nu(s_0)}$. Since $\mathcal{A}u \in H^{\nu(s)}$, we have

$$u = \mathfrak{R}\mathcal{A}u - Tu \in H^{\mu\nu(s_1)},$$

where $s_1 = \min(s, s_0 + \sigma)$. Iterating this argument, we prove the theorem.

Theorem 2 (on local regularity). Let the symbol $a(x,\xi)$ satisfy the conditions of Theorem 1 and Proposition 2. Then

$$\{u \in \mathcal{S}',\ \mathcal{A}u \in H^{\nu(s)}_{\mathrm{loc}}(\Omega) \Rightarrow u \in H^{\mu\nu(s)}_{\mathrm{loc}}(\Omega)\},$$

where Ω is a bounded domain in R^n.

Corollary.

$$\{u \in \mathcal{S}',\ \mathcal{A}u \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega)\},$$

i.e., a pseudodifferential operator satisfying the conditions of Theorem 2 is hypoelliptic.

Proof. If the domain Ω is bounded, then there exists such an s_0 that

$$u \in H^{\mu\nu(s_0)}_{\text{loc}}(\Omega).$$

Let $\varphi, \psi \in C_0^\infty(\Omega)$, with $\psi(x) = 1$ for $x \in \operatorname{supp}\varphi$. Then

$$\mathcal{A}(\varphi u) = \varphi\mathcal{A}u + [\mathcal{A}, \varphi]u = \varphi\mathcal{A}u + [\mathcal{A}, \varphi](\psi u) + \varphi\mathcal{A}((1-\psi)u). \tag{12}$$

By assumption, $\varphi\mathcal{A}u \in H^{\nu(s)}$; by Proposition 5,

$$[\mathcal{A}, \varphi](\psi u) \in H^{\nu(s_0+\sigma)};$$

and by Proposition 2 (pseudolocality),

$$\varphi\mathcal{A}((1-\psi)u) \in C_0^\infty(\Omega).$$

Thus the right-hand side of (12) belongs to $H^{\nu(s_1)}$, where

$$s_1 = \min(s, s_0 + \sigma).$$

By Theorem 1,

$$\varphi u \in H^{\mu\nu(s_1)},$$

i.e.,

$$u \in H^{\mu\nu(s_1)}_{\text{loc}}(\Omega).$$

Repeating this argument, we prove the theorem.

We shall make several concluding remarks.

1) The results of this work carry over trivially to systems of pseudodifferential operators constructed in the same way as hypoelliptic systems in ([12]).
2) Theorems 1 and 2 can be proved by the method of a priori estimates (see ([12,13])).
3) One may consider symbols that are regular only with respect to a group of variables, and construct a theory of partially hypoelliptic pseudodifferential operators generalizing the corresponding results (see ([12–14])) for differential operators.
4) As Yu. V. Egorov pointed out to the author, if the regularizer is constructed by a more refined method ([15]), then hypoellipticity can also be proved for pseudodifferential operators of "variable strength."

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Note: Figure translations are in progress. See original paper for figures.