ON THE THEORY OF FINITE FACTORIZABLE GROUPS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 519.413

MATHEMATICS

P. I. TROFIMOV

ON THE THEORY OF FINITE FACTORIZABLE GROUPS

(Presented by Academician A. I. Mal' cev, 28 VI 1965)

§ 1. A representation of a group in the form of a product (in the sense of multiplication of Frobenius complexes) of some of its subgroups, taken (in the case of more than two subgroups) in a definite order, is called, as is well known, a factorization of this group, and the subgroups themselves are called the factors of the factorization.

The principal problem posed for factorizable groups is the following:

To what extent do the structure and properties of the factors of a factorization of a group determine the structure and properties of this group itself?

Many papers by various authors have been devoted to the solution of this important problem for the case of a finite group (see, for example, the survey article ([1]); for later papers see, for example, ([2])). In most of these works criteria for solubility and nonsimplicity of finite factorizable groups were established, for which certain conditions were imposed on the factors of the factorization (Abelianness, nilpotency, and others).

In the present paper, for the first time, criteria are investigated for the solubility and nonsimplicity of finite groups factorizable by two of their proper subgroups, under the condition that at least one of the factors of the factorization is a group with one class of noninvariant conjugate subgroups.

§ 2. Let the finite group $\mathfrak{G}$ be factored by the proper subgroups $\mathfrak{A}$ and $\mathfrak{B}$.

No. 1. An example of a simple group of order 168 shows that the group $\mathfrak{G}$ may be simple in the case when the subgroup $\mathfrak{A}$ is nilpotent (even a p-group), and the subgroup $\mathfrak{B}$ contains only one class of noninvariant conjugate subgroups.

Indeed, the simple group of order 168 is factored by two of its proper subgroups of orders 8 and 21, and the second subgroup is non-Abelian and, consequently, contains only one class of noninvariant conjugate subgroups.

No. 2. The simple group of order 60 is factored by its proper subgroups $\mathfrak{A}$ and $\mathfrak{B}$ of orders 12 and 5, respectively. Analysis of the subgroup $\mathfrak{A}$ as a subgroup of the simple group of order 60 leads to the conclusion that $\mathfrak{A}$ is a group with two classes of noninvariant conjugate subgroups and that $\mathfrak{A}$ is a group of type S (the group itself is nonnilpotent, while all its proper subgroups are nilpotent). Consequently, among finite groups factored by two of their proper subgroups, one of which is a group with two classes of noninvariant conjugate subgroups or a group of type S, and the other Abelian (even cyclic of prime order), there exist simple groups.

§ 3. **Lemma 1.** *Let the finite group $\mathfrak{G}$ have the following properties:*

a) $\mathfrak{G}$ is factored by a nilpotent subgroup $\mathfrak{A}$ and a non-Abelian subgroup $\mathfrak{B}$ of order pq, where p and q are prime numbers and $p < q$; b) $\mathfrak{G}$ is insoluble.

Then $\mathfrak{A} \cap \mathfrak{B}$ coincides with the identity subgroup $\mathfrak{E}$.

Lemma 2. Let a finite group $\mathfrak{G}$ be factorized by its proper subgroups $\mathfrak{H}$ and $\mathfrak{H}_1$, and suppose that the intersection of the subgroups $\mathfrak{H}$ and $\mathfrak{H}_1$ is different from the identity and contains a normal divisor of the subgroup $\mathfrak{H}_1$. Then the group $\mathfrak{G}$ has a proper normal divisor contained in the subgroup $\mathfrak{H}$ (Lemma 1 of paper ([3])).

Lemma 3. In a finite group with one class of noninvariant conjugate subgroups and of order $p^\alpha q$ (p, q are primes, $p < q$), every subgroup of order $p^\beta q$ ($0 \leq \beta < \alpha$) is cyclic and invariant in this group.

Lemma 4. The dihedral group of order 8 is a group with two classes of noninvariant conjugate subgroups.

Lemma 5. Every factor group of a group with k ($k \geq 1$) classes of noninvariant conjugate subgroups contains no more than k classes (of noninvariant conjugate subgroups).

Lemma 6. The simple group of order 60 is not factorized by two of its proper subgroups each of which contains one class of noninvariant conjugate subgroups.

§ 4. **Theorem 1.** Let $\mathfrak{G}$ be a finite group factorized by a nilpotent subgroup $\mathfrak{A}$ and a subgroup $\mathfrak{B}$ containing no more than one class of noninvariant conjugate subgroups. Suppose, moreover, that $\mathfrak{A} \cap \mathfrak{B} \neq \mathfrak{E}$. Then the group $\mathfrak{G}$ contains a proper solvable normal divisor.

Theorem 2. A finite group is solvable if it is factorized by a nilpotent subgroup and a subgroup of odd order containing only one class of noninvariant conjugate subgroups.

Remark. Among finite groups factorized by a nilpotent subgroup and a subgroup of odd order and with one class of noninvariant conjugate subgroups, there exist simple ones.

Indeed, the simple group of order 168 is factorized by a nilpotent subgroup (of order 8) and a subgroup of order 21 with one class of noninvariant conjugate subgroups.

Theorem 3. A finite group factorized by a subgroup with one class of noninvariant conjugate subgroups and of order pq (p and q are primes, $p < q$) and by a nilpotent subgroup whose Sylow subgroup for the smallest prime divisor of its order is abelian or a quaternion group, is solvable.

Theorem 4. A finite group factorized by an abelian or Hamiltonian subgroup and a subgroup with one class of noninvariant conjugate subgroups is solvable.

Theorem 5. A finite group is solvable if it is factorized by two of its proper subgroups each of which contains one class of noninvariant conjugate subgroups.

In proving our theorems, in addition to the lemmas, we used: the theorems obtained here by us, some of the known criteria for solvability and nonsimplicity of finite factorizable groups, as well as other theorems, in particular the Feit–Thompson theorem ([4]) on the solvability of every finite group of odd order.

For the proofs of Theorems 4 and 5 we used the method of mathematical induction (in combination with the indirect method of proof). In proving Theorem 5 we also had to apply certain other devices.

Perm State University
named after A. M. Gorky

Received
5 VI 1965

CITED LITERATURE

[1] S. A. Chunikhin, UMN, **17**, no. 4, 31 (1961).
[2] O. H. Kegel, Arch. Math., **12**, no. 2, 90 (1961).
[3] S. A. Chunikhin, *Proceedings of the Seminar on the Theory of Finite Groups*, Moscow-Leningrad, 1938, p. 106.
[4] W. Feit, J. G. Thompson, Pacific J. Math., **13**, no. 3, 771 (1963).

Note: Figure translations are in progress. See original paper for figures.