ON A PROBLEM OF I. M. GELFAND

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Abstract

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ON A PROBLEM OF I. M. GELFAND

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Let $f(x) = f(x_1, \dots, x_n)$ be a finite infinitely differentiable function in a real n-dimensional affine space, which we shall assume to be oriented, let $P = P(x) = \sum_{s,t=1}^{n} a_{st}x_{sx}t$ be a nondegenerate quadratic form having, in its canonical representation, p positive and q negative squares ($p + q = n$), and let $Q = Q(\sigma) = \sum_{s,t=1}^{n} a^{st}\sigma_s\sigma_t$ be the quadratic form dual to P (a^{st} are the elements of the matrix inverse to $\|a_{st}\|$). To the function $f(x)$ we associate its integrals over those hyperplanes $(x, \xi) = u$ for which $Q(\xi) > 0$:

$$\check{f}(\xi, u) = \int f(x)\delta(u - (x, \xi))\, dx,$$

and also its integrals over the second-order surfaces $P(x, y) + u = 0$ for $u > 0$:

$$\check{f}_P(y, u) = a \int (x + y)\delta(u + P(x))\, dx.$$

Here δ is the delta-function, $dx = dx_1, \dots, dx_n$, the integration is over the whole space, and $a = \pi^{-n/2}|\det \|a_{st}\||^{1/2}$.

It is required, knowing $\check{f}(\xi, u)$ and $\check{f}_P(y, u)$, to reconstruct the original function $f(x)$.

The formulation of this problem of integral geometry in terms of Minkowski space ($p = 1$) is due to I. M. Gelfand (^{1}); there its solution is also given in the case of three-dimensional space. For odd q, in order to reconstruct $f(x)$ it is sufficient to know only $\check{f}_P(y, u)$, $u > 0$:

$$f(x) = (-1)^{p/2}\sqrt{\pi} \lim_{u \to 0} \sqrt{u}\, \frac{\partial^{(n-1)/2} \check{f}_P(x, u)}{\partial u^{(n-1)/2}} \tag{1}$$

for a space of odd dimension,

$$f(x) = (-1)^{(p+1)/2}\pi \lim_{u \to 0} u\, \frac{\partial^{n/2} \check{f}_P(x, u)}{\partial u^{n/2}}, \tag{2}$$

if n is even. Formulas (1) and (2) can be derived from the general results of $(^2)$, § 4, Ch. III. Therefore it is sufficient to consider the case of even q. In the present work the problem is solved by the Fourier-transform method for arbitrary p and even q.

First we formulate the results obtained. For even q, if p is odd, the relation holds

$$f(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{Q(\xi)>0} \left.\frac{\partial^{n-1} \check{f}(\xi, u)}{\partial u^{n-1}}\right|_{u=(x,\xi)} \omega(\xi) + \frac{(-1)^{(p+1)/2}}{\sqrt{\pi}} \int_0^{\infty} c^{-1/2} \frac{\partial^{(n-1)/2} \check{f}_P(x, c)}{\partial c^{(n-1)/2}}\, dc. \qquad (3)$$

If p and q are even, then

$$f(x) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n} \int_{Q(\xi)>0} \omega(\xi) \int_{-\infty}^{+\infty} u^{-n} f(\xi, u + (x, \xi))\, du + + (-1)^{p/2} \lim_{u\to 0+0} \frac{\partial^{n/2-1} f_P(x, u)}{\partial u^{n/2-1}}. \qquad (4)$$

Here

$$\omega(\xi) = \sum_{k=1}^{n} (-1)^{k-1} \xi_k\, d\xi_1 \dots d\xi_{k-1} d\xi_{k+1} \dots d\xi_n,$$

integration with respect to ξ is carried out over the set of all rays with direction vector ξ, issuing from the point $\xi = 0$, for which $Q(\xi) > 0$ (the integrands are constant along such rays). The integral with respect to the parameter c in (3) converges in the ordinary sense. The integral with respect to u in the first term of (4) is understood in the sense of a regularized value, namely, as the analytic continuation in λ of the function

$$\int_{-\infty}^{+\infty} |u|^{\lambda} \tilde{f}(\xi, u + (x, \xi))\, du,$$

regular at the point $\lambda = -n$ (n is an even number). This regularization is effected by the formula

$$\int_{-\infty}^{+\infty} u^{-n} \psi(u)\, du = \int_0^{\infty} u^{-n} \left\{ \psi(u) + \psi(-u) - 2 \left[\psi(0) + \frac{u^2}{2} \psi''(0) + \dots \right.\right.$$

$$... + \frac{u^{n-2}}{(n-2)!}\psi^{(n-1)}(0)\Big]\Big]\Big\}\, du,$$

where $\psi(u)$ is a finite differentiable function of one variable $^{(2)}$.

Let us note that, for a positive definite form $P(x)$, (3) and (4) pass into Radon' s formulas for the expansion of a function into its plane waves in affine-invariant notation; in this case the second terms vanish, and integration with respect to ξ is carried out over all rays issuing from the origin.

We give a sketch of the proof, restricting ourselves to the case when $P(x)$ has, in its canonical representation, an even number of positive and an even number of negative squares. In what follows, a wavy line over a functional denotes its Fourier transform. Consider the generalized function

$$\frac{|(x,\xi)|^{\lambda}}{\Gamma\left(\frac{\lambda+1}{2}\right)},$$

where λ is a complex parameter. Relying on the analogous result for one independent variable $^{(2)}$, one can show that the Fourier transform of this functional is the functional acting in the corresponding dual space of functions $\varphi(\sigma) = \varphi(\sigma_1, ..., \sigma_n)$ according to the formula

$$\frac{2^{\lambda+n}\pi^{n-1/2}}{\Gamma(-\lambda/2)}\int_{-\infty}^{+\infty}|t|^{-\lambda-1}\varphi(t\xi)\, dt.$$

Putting then $\lambda = -n$ (n is an even number), after multiplication by $\omega(\xi)$ and integration with respect to ξ, we obtain that the Fourier transform of the generalized function

$$F_1 = \frac{(-1)^{n/2}(n-1)}{(2\pi)^n}\int_{Q(\xi)>0}(x,\xi)^{-n}\omega(\xi)$$

is equal to the characteristic function of the domain $Q(\sigma) > 0$. Next, starting from the equality

$$P_-^{\lambda} = \int_0^{\infty} c^{\lambda}\delta(c+P)\, dc,$$

where

$$P_-^{\lambda} = \begin{cases}(-P)^{\lambda}, & \text{for } P < 0,\\ 0, & \text{for } P \geqslant 0\end{cases}$$

(similar notation is also used below), which for $\operatorname{Re}\lambda < 0$ is understood in the sense of analytic continuation of the corresponding functionals in λ, after integrating the integral by parts $n/2$ times; for $\lambda = -n$ we obtain (P_-^λ is known from (2)) that the Fourier transform of the functional

$$F_2 = -(-1)^{p/2} a \int_0^\infty \delta^{(n/2)}(c+P)\,dc = (-1)^{p/2} a \lim_{c\to 0+0} \delta^{(n/2-1)}(c+P)$$

is equal to the characteristic function of the domain $Q(\sigma) < 0$. Thus, $F_1 + F_2 = 1$. Since the inverse Fourier transform of the unit functional is $\delta = \delta(x_1, \dots, x_n)$, we have $\delta = F_1 + F_2$, which is equivalent to (4).

Formulas (1) and (2) can also be proved with the aid of the Fourier transform. For example, if n is odd and p is even, then the Fourier transform of $\delta^{(n-1)/2}$, $u > 0$, is equal to

$$\frac{(-1)^{p/2}}{a\sqrt{u\pi}} \left[\exp\left(-\sqrt{uQ_+}\right) + \cos\sqrt{uQ_-}\right],$$

where the exponential function and the cosine are generalized functions defined by means of the formal expansion into power series (2), whence

$$\delta = (-1)^{p/2} a\sqrt{\pi} \lim_{u\to 0} \sqrt{u}\,\delta^{((n-1)/2)}(u+P),$$

which is equivalent to (1)*.

In conclusion I express my gratitude to M. I. Graev for discussion and valuable remarks.

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REFERENCES

[1] I. M. Gelfand, *UMN*, vol. 2 (92) (1960).
[2] I. M. Gelfand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1959.

* **Note added in proof.** As became known to the author after the note had been submitted to the editorial office, formula (3) was obtained by V. I. Semyanistyi under additional orthogonality conditions imposed on the function.

Note: Figure translations are in progress. See original paper for figures.

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