# DOUBLE OPERATOR STIELTJES INTEGRALS AND MULTIPLIER PROBLEMS

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## Abstract

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*MATHEMATICS*

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# DOUBLE OPERATOR STIELTJES INTEGRALS AND MULTIPLIER PROBLEMS

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In a note by the authors ($^1$) (see also ($^2$)), properties were studied of double operator Stieltjes integrals of the form

$$Q \equiv \Phi T = \iint_{\Lambda M} \varphi(\lambda, \mu)\, F(d\mu)\, T E(d\lambda), \tag{1}$$

where $\varphi$ is a complex function, $T$ is a bounded operator in a Hilbert space $\mathfrak{H}$, and $F(\cdot)$ and $E(\cdot)$ are orthogonal spectral measures. Here we shall present further results in this direction, using the notation and terminology of ($^1$).

In Sec. 1 the properties are studied of the transformation (transformer) $Q = \Phi T$, acting in various symmetric normed ideals ($^3$) of the ring $\mathfrak{R}$ of bounded operators in $\mathfrak{H}$. The corresponding results of ($^1$) are generalized by the fact that, first, we no longer assume $\Lambda$ and $M$ to be one-dimensional manifolds and, second, the smoothness conditions on the function $\varphi$ are imposed not in the uniform metric but in the metric of the space $L_p$. Such generalizations are essential for applications.

In Sec. 2 the connection of integrals of the form (1) with a certain class of multiplier problems is clarified. In particular, our scheme includes the question of the boundedness of multidimensional singular integrals in $L_2$ (Sec. 3) and the trigonometric multiplier problem in $l_p$ (Sec. 4). From the general theorems on transformers there follow both certain known and new results for these problems. Let us also note that, by writing a singular integral in the form (1), we give for it (apparently for the first time) an invariant definition.

1. Let $(\Lambda, \Sigma, E(\cdot))$, $(M, \Sigma_1, F(\cdot))$ be two spaces with orthogonal spectral measures. The definition of the integral (1), given in ($^1$) for $\Lambda = M = R^1$, carries over automatically to the case under consideration. Without dwelling on this question in detail, we point out only that, as in ($^1$), one can introduce a certain special orthogonal measure $\mathfrak{G}(\cdot)$ (essentially, the tensor product of the measures $E(\cdot)$ and $F(\cdot)$). The transformer $\Phi$ is then defined as the integral of the function $\varphi(\lambda, \mu)$ with respect to the measure

$\mathfrak{G}(\cdot)$. Below we shall assume the function $\varphi(\lambda, \mu)$ to be measurable and bounded with respect to the measure $\mathfrak{G}$:

$$(\mathfrak{G}) \;-\; \sup |\varphi(\lambda, \mu)| < \infty. \tag{2}$$

The analogue of Theorem 1 from ($^1$) is the following proposition.

**Theorem 1.** *Under condition (2), the integral (1) defines a transformer* $\Phi$ *of class* $(\mathfrak{S}_2, \mathfrak{S}_2)$. *The set of such transformers forms a commutative normed ring with involution, isomorphic and isometric to the ring of* $\mathfrak{G}$*-measurable functions with norm (2).*

Denote by $Q^m$ the unit cube in $R^m$, and by $W_p^\alpha(Q^m)$ the corresponding Sobolev-Slobodetskii functional space ($^4$).

For $m = 1$ we shall also consider the space $V_\beta$ (see, for example, ($^5$)) of functions of bounded $\beta$-variation.

**Theorem 2.** Let $\Lambda = Q^m$, $\varphi(\lambda, \mu) \in W_p^\alpha(Q^m)$ $(\alpha p > m)$ as a function of $\lambda$ for almost all $\mu \in M$, and suppose that

$$(F) \text{ —} \sup \|\varphi(\cdot, \mu)\|_{W_p^\alpha} < \infty. \tag{3}$$

Then, for $p \leqslant 2$, the integral (1) generates a transformer belonging to each of the classes $(\mathfrak{S}_q, \mathfrak{S}_q)$, $1 \leqslant q \leqslant \infty$, and $(\mathfrak{R}, \mathfrak{R})$. For $m = 1$ this result is also true if, for almost all $\mu \in M$, $\varphi \in V_\beta$, $\beta < 2$, and $\varphi \in \operatorname{Lip} \varepsilon$, $\varepsilon > 0$, with respect to the variable $\lambda$, and

$$(F) \text{ —} \sup\{\|\varphi(\cdot, \mu)\|_{\operatorname{Lip} \varepsilon} + \|\varphi(\cdot, \mu)\|_{V_\beta}\} < \infty. \tag{4}$$

Analogous assertions are valid when the roles of the variables $\lambda$ and $\mu$ are interchanged.

We note that for $m = 1$ condition (4) is broader than condition (3).

**Corollary 1.** Under the hypotheses of Theorem 2, $\Phi \in (\mathfrak{S}, \mathfrak{S})$, where $\mathfrak{S}$ is any separable ($^3$) (or conjugate to a separable) symmetric normed ideal in $\mathfrak{R}$.

**Corollary 2.** Under the hypotheses of Theorem 2, the transformer $\Phi$ induces on the factor-ring $\mathfrak{R}/\mathfrak{S}_\infty$ a bounded linear transformation whose norm does not exceed $\|\Phi\|_{1,1}$.

**Theorem 3.** Suppose that the hypotheses of the first part of Theorem 2 are satisfied, $p > 2$, and $\alpha \leqslant m/2$. Then the integral (1) generates a transformer $\Phi \in (\mathfrak{S}_q, \mathfrak{S}_q)$, where $|q^{-1} - 1/2| < \alpha m^{-1}$. If $\alpha = m/2$, then $\Phi \in (\mathfrak{S}_1, \mathfrak{S}_\infty)$ and $\Phi \in (\mathfrak{S}_\infty, \mathfrak{S}_\infty)^*$. If, in the hypotheses of the second part of Theorem 2, $\beta \geqslant 2$, then $\Phi \in (\mathfrak{S}_q, \mathfrak{S}_q)$ for $|q^{-1} - 1/2| < \beta^{-1}$.

**Remark.** The assertions of Theorems 2 and 3 remain valid if $\Lambda$ is any smooth compact $m$-dimensional manifold, without boundary or with boundary.

We shall give brief indications concerning the method of proof of Theorems 2 and 3. Let $f, g \in \mathfrak{H}$, $\|f\| = \|g\| = 1$, and let the measures $\sigma$ and $\tau$ be given by the formulas $\sigma(\cdot) = (E(\cdot)f, f)$, $\tau(\cdot) = (F(\cdot)g, g)$. According to the scheme set forth in $({}^2)$, to prove Theorem 2 it suffices to estimate, uniformly with respect to $f$ and $g$, the nuclear norm of the integral operator with kernel $\varphi(\lambda, \mu)$ acting from $L_2(\Lambda; \sigma)$ into $L_2(M; \tau)$. The required uniform estimates were obtained in the authors' paper $({}^5)$. The proof of Theorem 3 uses the general idea of "interpolation by smoothness," applied earlier by Khirman $({}^6)$ in a simpler situation. In carrying out the interpolation, it is necessary to make essential use of a number of results of $({}^5)$, including Theorem 1 of $({}^5)$, as well as the concrete method described there for approximating functions from $W_p^\alpha$ by piecewise polynomial functions. The proof of Corollary 1 is based on the interpolation theorem of B. S. Mityagin $({}^7)$.

2. Let the space $\mathfrak{H}$ be decomposed into a direct integral

$$\mathfrak{H} = \int_\Lambda \oplus \mathfrak{H}(\lambda)\rho(d\lambda) \qquad \left(\mathfrak{H} = \int_{\widetilde{M}} \oplus \mathfrak{H}_1(\mu)\rho_1(d\mu)\right) \tag{5}$$

so that the action of the operator $E(\delta)$, $\delta \subset \Lambda$ $(F(\partial), \partial \subset M)$, on an element $f \in \mathfrak{H}$ reduces to multiplication of the "representative" $f(\lambda)$ $(f_1(\mu))$ by the characteristic function of the set $\delta(\partial)$:

$$(E(\delta)f)(\lambda) = \chi_\delta(\lambda)f(\lambda) \qquad ((F(\partial)f)_1(\mu) = \chi_\partial(\mu)f_1(\mu)).$$

To every operator $A \in \mathfrak{S}_2$, in the decompositions (5) there corresponds an operator kernel $A(\lambda, \mu)$, determined almost everywhere on $\Lambda \times M$, of class $\mathfrak{S}_2$, mapping $\mathfrak{H}(\lambda)$ into $\mathfrak{H}_1(\mu)$ and such that

$$(Af)_1(\mu) = \int_\Lambda A(\lambda, \mu)f(\lambda)\rho(d\lambda),$$

$$\|A\|^2_{\mathfrak{S}_2} = \iint_{\Lambda M} \|A(\lambda, \mu)\|^2_{\mathfrak{S}_2}\rho(d\lambda)\rho_1(d\mu).$$

It is easy to verify that, under the hypotheses of Theorem 1, the kernels corresponding to the operators $T$ and $Q$ are related by

* For the definition of the ideals $\mathfrak{S}_\omega, \mathfrak{S}_\Omega$ see $({}^3)$, §15.

$$Q(\lambda, \mu) = \varphi(\lambda, \mu) T(\lambda, \mu).$$

Denote by $\widetilde{\mathfrak{S}}_q$, $q \leq 2$, the class of kernels corresponding, in the decompositions (5), to operators of the class $\mathfrak{S}_q$. Starting from the relation $\mathfrak{S}_1^* = \mathfrak{R}$, one can associate a generalized kernel with every operator $A \in \mathfrak{R}$. Thus we arrive at the classes of generalized kernels $\widetilde{\mathfrak{R}}$, $\widetilde{\mathfrak{S}}_q$, $2 < q \leq \infty$, associated with the decompositions (5). The assertions of Theorems 2 and 3 can now be regarded as sufficient conditions for a scalar kernel $\varphi(\lambda, \mu)$ to be a multiplier in some of the classes $\widetilde{\mathfrak{S}}_q, \widetilde{\mathfrak{R}}$.

3. In this section $\mathfrak{H} = L_2(R^m)$, $\Lambda = R^m$, $M = S_{m-1}$ is the unit sphere in the space $\Xi_m$ dual to $R^m$. If $\delta \subset S_{m-1}$, then $\bar{\delta}(\subset \Xi_m)$ is the complete inverse image of $\delta$ under central projection. Let $F(\partial)$ be the operator of multiplication by the characteristic function of a set $\partial \subset R^m$, let $\mathcal{F}$ be the Fourier operator, and let $E(\delta) = \mathcal{F}^* F(\bar{\delta}) \mathcal{F}$, $\delta \subset S_{m-1}$. Suppose further that a function $\sigma(\theta, x)$ $(\theta \in S_{m-1}, x \in R^m)$ defines the transformer

$$\Phi = \iint \sigma(\theta, x) F(dx)(\cdot) E(d\theta)$$

of the class $(\mathfrak{R}, \mathfrak{R})$. We introduce the following definition.

**Definition.** A **singular integral (s.i.) operator** in $L_2(R^m)$ with symbol $\sigma(\theta, x)$ is the operator $J = \Phi I$ ($I$ is the identity operator).

The considerations of § 2 make it possible easily to establish the equivalence of this definition and the customary definitions of an s.i. operator in terms of its symbol by means of the Fourier transform ($^{8-10}$) or by means of a certain repeated integral ($^{8}$). The advantage of the definition proposed here is its invariance and symmetry with respect to $\theta$ and $x$.

Theorem 2 now leads to various criteria for boundedness of s.i. operators in $L_2(R^m)$. We shall agree to write $\sigma \in \widetilde{W}_p^\alpha(S_{m-1})$ if $\sigma(\theta, x)$ belongs to $W_p^\alpha$ with respect to $\theta$, and the norm of $\sigma$ as a function of $x$ is essentially (with respect to Lebesgue measure) bounded. Analogously for other functional classes, and also when the roles of $\theta$ and $x$ are interchanged.

**Theorem 4.** Let $J$ be an s.i. operator with symbol $\sigma(\theta, x)$, where $\sigma \in \widetilde{W}_2^\alpha(S_{m-1})$, $2\alpha > m - 1$, or, for $m = 2$, $\sigma \in \widetilde{V}_\beta(S_1) \cap \operatorname{Lip} \varepsilon(S_1)$, $\beta < 2$, $\varepsilon > 0$. Then $J \in \mathfrak{R}$.

The first assertion of Theorem 4 was recently obtained, with the aid of expansions in spherical functions, by M. S. Agranovich ($^{9}$), who refined the considerations of S. G. Mikhlin ($^{8}$). The second assertion is apparently new.

Interchanging the roles of the variables $x$ and $\theta$ leads to another type of boundedness conditions. Let $x = x(y)$ be the inverse stereographic mapping of the sphere $S^m$ onto the extended space $R^m$.

**Theorem 5.** Let $J$ be an s.i. operator with symbol $\sigma(\theta, x)$ and $\hat{\sigma}(\theta, y) = \sigma(\theta, x(y))$. If $\hat{\sigma} \in \widetilde{W}_2^{\alpha}(S^m)$, $2\alpha > m$, then $J \in \mathfrak{R}$.

If the symbol $\sigma(\theta, x)$ is finite with respect to $x$, then the assertion of Theorem 5 is valid under the condition $\sigma \in \widetilde{W}_2^{\alpha}(R^m)$, $2\alpha > m$. Imposing the latter condition locally, we arrive at a boundedness criterion for an s.i. operator in $L_2$ on a compact $m$-dimensional manifold.

The approach described makes results such as Theorem 5 quite natural. At the same time, until very recently the possibility of imposing requirements on the symbol with respect to the variable $x$, and not to $\theta$, had not been noticed at all. Only recently, in a paper of Kohn and Nirenberg [10], was a similar condition proposed in terms of the Fourier transform of the symbol with respect to the variable $x$. This condition and Theorem 5 do not cover one another, but, when formulated in terms of smoothness of the symbol, our conditions are less restrictive.

4. Let $\{c_k\}$, $k = (k_1, \dots, k_m)$, $-\infty < k_j < \infty$, be an $m$-fold sequence of class $l_q$, $q \geqslant 1$. Multiply the function

$$f(\lambda) = \sum_k c_k e^{i(k,\lambda)}, \qquad \lambda = (\lambda_1, \dots, \lambda_m)$$

(for $q > 2$, possibly generalized) by a fixed periodic function $\psi(\lambda)$. Let $\Psi$ be the linear transformation which sends the sequence $\{c_k\}$ to the sequence of Fourier coefficients of the function $\psi(\lambda) f(\lambda)$. It is required to indicate conditions under which $\Psi$ is a bounded operator from $l_q$ to $l_q$. The formulated trigonometric problem of multipliers [11,6] is included in the scheme of § 2 if one sets $\Delta = M = \mathcal{T}_m$, where $\mathcal{T}_m$ is the $m$-dimensional torus; $\mathfrak{H} = L_2(\mathcal{T}_m)$; $E(\delta) = F(\delta)$ is the operator of multiplication by the characteristic function of the set $\delta \subset \mathcal{T}_m$; $T$ is the integral operator in $L_2(\mathcal{T}_m)$ with kernel $f(\mu - \lambda)$; $\varphi(\lambda, \mu) = \psi(\mu - \lambda)$. Application of Theorems 2 and 3 leads to the following result.

**Theorem 6.** If $\psi \in W_p^{\alpha}(\mathcal{T}_m)$, $p\alpha > m$, then $\Psi \in (l_q, l_q)$, where $q \geqslant 1$ for $p \leqslant 2$ and $|q^{-1} - 1/2| < \alpha m^{-1}$ for $p > 2$, $2\alpha \leqslant m$.

As for the second assertion of Theorem 3, in the present case it leads to a result obtained earlier by Hirschman [6] on another technical basis.

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## REFERENCES

[1] M. Sh. Birman, M. Z. Solomyak, DAN, **165**, No. 6 (1965).
[2] M. Sh. Birman, M. Z. Solomyak, in the collection *Problems of Mathematical*

*Physics*, issue 1, L., 1966.

[3] I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Non-Selfadjoint Operators*, "Nauka," 1965.

[4] S. M. Nikol' skii, UMN, **16**, issue 5 (101) (1961).

[5] M. Sh. Birman, M. Z. Solomyak, DAN, **171**, No. 5 (1966).

[6] I. I. Hirschman, Duke Math. J., **26**, No. 2 (1959).

[7] B. S. Mityagin, Matem. sborn., **66** (108), 4 (1965).

[8] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, M., 1962.

[9] M. S. Agranovich, UMN, **20**, issue 5 (125) (1965).

[10] J. J. Kohn, L. Nirenberg, Comm. Pure and Appl. Math., **18**, No. 1/2 (1965).

[11] S. B. Stechkin, DAN, **71**, No. 3 (1950).

*Note: Figure translations are in progress. See original paper for figures.*

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