THE METHOD OF GRIDS FOR FINITE AND INFINITE DOMAINS WITH PIECEWISE SMOOTH BOUNDARY

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Abstract

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MATHEMATICS

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THE METHOD OF GRIDS FOR FINITE AND INFINITE DOMAINS WITH PIECEWISE SMOOTH BOUNDARY

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1. In the work (1), N. S. Bakhvalov gave an estimate of the error of the grid solution of the Dirichlet problem for Laplace' s equation in a fairly broad class of finite domains, of order $h^2 \ln h^{-1}$, where h is the mesh size of the square grid, under the assumption that the desired solution has bounded second derivatives. However, if the boundary of the domain has an angle $\alpha\pi$, $\alpha > 1/2$, $\alpha \neq 1$, then, as a rule, the solution has unbounded derivatives of order $[1/\alpha] + 1$ (see (2–4)). Moreover, one can give an example of a Dirichlet problem in a finite domain with piecewise analytic boundary having an angle $\alpha\pi$, $1/2 < \alpha \leq 2$, $\alpha \neq 1$, with boundary conditions continuous on the whole boundary and analytic at points of analyticity of the boundary, for which the approximate solution obtained on a square grid by the method of (5) has a maximum error bounded below, for $h < h_\alpha$, by the quantity $c_\alpha h^{1/\alpha}$, where $c_\alpha > 0$. In (6), P. Laasonen considers a finite domain with piecewise analytic boundary having a finite number of angles $\alpha_j\pi$, $0 < \alpha_j < 1$. Under the assumption that the boundary function is continuously differentiable 5 times inside the analytic arcs and continuous at the corner points, (6) obtains an estimate of order h^2 for the error of the grid solution of the Dirichlet problem on any subdomain obtained by discarding finite neighborhoods of the corner points. In (7), the author proposed, for the solution of the Dirichlet problem on finite polygons, two difference schemes on a uniform square grid, of orders of accuracy $h^{2-\varepsilon}$ and $h^{6-\varepsilon_1}$, where ε and ε_1 can be made arbitrarily small positive quantities at the cost of increasing the order of the required bounded derivatives of the boundary function and complicating the schemes.

In the present work we consider a difference method, simpler and more universal than in (7), for solving the Dirichlet problem for Laplace' s equation on a certain class of, generally speaking, infinite domains with piecewise smooth boundary having a finite number of angles $\alpha_j\pi$, $0 \leq \alpha_j \leq 2$. For a boundary function which, on the pieces of the boundary between the corner points, has a Hölder-continuous (decreasing sufficiently rapidly at infinity) third derivative, discontinuities of the first kind are allowed at the corner points. A composite grid is constructed. In a finite neighborhood of corner points lying in the bounded part of the plane, a polar grid is constructed, becoming denser as it approaches the vertex of the angle. On infinite branches of the domain, a coarsening polar grid is constructed. In addition, a base is constructed—a square grid with mesh size h, covering the finite part of the domain and intersecting all the polar grids. The total number of grid nodes has order $h^{-2}\ln h^{-1}$. The system of difference equations obtained on the composite grid has a unique solution converging uniformly to the bounded solution of the Dirichlet problem with rate $h^2 \ln h^{-1}$. The difference operators used at the interior nodes of the polar grid, as well as on the square grid, prescribe the desired

value as the arithmetic mean of the values of the function at the four neighboring nodes.

2. Let a simply connected domain G be given with boundary γ, consisting of N pieces (sides) γ_j, $j = 1, 2, \dots, N$, numbered counterclockwise, having differentiable curvature. The sides may be unbounded, but for simplicity we shall assume that the unbounded branches of the sides have asymptotes. The derivative of the curvature of γ_j satisfies a Hölder condition with constant and positive exponent independent of the choice of points on γ_j, and decreases at infinity as R^{-4}, where R is the distance from the origin. Let $\alpha_j\pi$, $0 \le \alpha_j \le 2$, be the interior angle formed by the tangents (asymptotes) to γ_{j-1} ($\gamma_0 \equiv \gamma_N$) and γ_j at the vertex (at the point of intersection). To each angle we assign a point Q_j and an index σ_j. If the vertex of the angle lies in the finite part of the plane, then Q_j coincides with the vertex and $\sigma_j = 1$. Otherwise $\sigma_j = -1$, and for $\{\alpha_j\} > 0$ Q_j is the point of intersection of the asymptotes, while for $\{\alpha_j\} = 0$ the point Q_j is located at equal distance from the asymptotes. If $\alpha_j = 0$, $\sigma_j = 1$, then it is assumed that γ_{j-1} and γ_j have different curvature at Q_j and, moreover, in a finite neighborhood of Q_j the curvature of the sides has a second derivative satisfying a Hölder condition. If $\alpha_j = 0$, $\sigma_j = -1$, then the asymptotes to γ_{j-1} and γ_j do not coincide. For $\sigma_j = -1$, the distance ρ_{jp} from γ_{j-p} to the asymptote in a neighborhood of the vertex of the angle $\alpha_j\pi$ has the expression

$$\rho_{jp} = \sum_{k=1}^{m_j-1} a_{jp}^k r_j^{-k} + \rho_{jp}^*, \quad \text{where } r_j \text{ is the distance from } Q_j; \quad m_j = [3-\alpha_j]; \quad \rho_{jp}^*$$

$$= O(r_j^{-m_j}) \quad \text{and, moreover, for } \alpha_j = 0 \quad \rho_{jp}^{*(1)}(r_j) = O(r_j^{-3}); \quad p = 0, 1; \quad a_{jp}^k \text{ are numbers.}$$

Consider the Dirichlet problem

$$\Delta u = 0 \ \text{ in } G, \qquad u = \varphi_j \ \text{ on } \gamma_j, \qquad j = 1, 2, \dots, N, \tag{1}$$

where $\varphi_j = \varphi_j(s)$ is a function of the arc s, continuous on γ_j (including the endpoints), having a third derivative, and

$$|\varphi_j(s)| \le \Phi_j; \qquad |\varphi_j^{(m)}(s)| \le \Phi_j/(r_j+1)^{m+1},$$

$$m = 1, 2, 3; \qquad |\varphi^{(3)}(s) - \varphi_j^{(3)}(s_1)| \le \Phi_j |s - s_1|^{\lambda_j}, \qquad \lambda_j > 0,$$

where s, s_1 are arbitrary points on γ_j; Φ_j, λ_j are constants; $j = 1, 2, \dots, N$. For $\sigma_j = -1$, in a neighborhood of the vertex of the angle $\alpha_j \pi$, the function φ_{j-p} has the form

$$\varphi_{j-p} = \sum_{k=0}^{m_j-1} b_{jp}^k r_j^{-k} + \varphi_{jp}^*, \qquad \text{where } \varphi_{jp}^* = O(r_j^{-m_j})$$

and, moreover, for $\alpha_j = 0$

$$\varphi_{jp}^{*(1)}(r_j) = O(r_j^{-3}); \qquad p = 0, 1; \quad b_{jp}^k \text{ are numbers.}$$

Coincidence of the values of φ_{j-1} and φ_j at the vertex of the angle $\alpha_j \pi$ is not required, $j = 1, 2, \dots, N$.

Construct N curvilinear triangles $g_j \subset G$, formed by the sides γ_{j-1}, γ_j and by arcs of circles of radii r_{j0} with centers at the points Q_j. The radii r_{j0} are chosen so that $g_k \cap g_m = \emptyset$ for $k \ne m$, and the tangent to γ_j (γ_{j-1}) at any point of a side of the triangle forms with the radius vector drawn from the point Q_j an angle not exceeding $\pi/3$.

Construct a square grid of straight lines $x, y = 0, \pm h, \pm 2h, \dots$. The set of grid nodes belonging to $\overline{G}$, together with the segments of straight lines joining them to the four nearest nodes and lying at a distance not exceeding h from

$$G \setminus \bigcup_{j=1}^{N} g_j$$

will be denoted by G_h. The set of nodes belonging to g_j and lying at distance h from G_h will be denoted by G_{jh}, $j = 1, 2, \dots, N$. The set of the remaining nodes lying at distance h from G_h will be denoted by γ_h. On g_j construct a polar grid with pole at the point Q_j, formed by the rays $\theta = \beta_j, 2\beta_j, \dots, \nu_j \beta_j$, $\nu_j = [2\pi r_{j0}/h] + 1$, $\beta_j = 2\pi/\nu_j$, and $N_j + 1$ circles of radii $r_{jk} = r_{j0}(1 + h_j)^{-k\omega_j}$, $k = 0, 1, \dots, N_j$; $h_j = 2(\sqrt{1 + \sin^2(\beta_j/2)} +$ $+ \sin(\beta_j/2))\sin(\beta_j/2)$. For $\alpha_j > 0$, $N_j = [\chi_j h_j^{-1}(c_j + \ln h_j^{-1})]$, $\chi_j = 2\max\{1, \alpha_j\}$, c_j is a constant. For $\alpha_j = 0$, N_j is a quantity of order $h^{-1} \ln h^{-1}$ and is chosen so that on the circle of radius r_{jN} there is no more than one node belonging to g_j, while on any circle of radius r_{jk}, where $k < N_j$, there is more than one node. The set of nodes of the polar grid lying on g_j and having the property that the four neighboring nodes of this grid belong to $\bar{g}_j$ will be denoted by g_{jh}. The set of nodes lying on the circle of radius r_{j0} will be denoted by g^0_{jh}. The set of the remaining nodes of the polar grid belonging to $\bar{g}_j$ will be denoted by γ_{jh}.

Introduce on G_h the averaging operator A,

$$Au(x,y) \equiv (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/4.$$

On g_{jh} construct the averaging operator A_j,

$$A_j\tilde{u}(r,\theta) \equiv (\tilde{u}(r(1+h_j),\theta) + \tilde{u}\left(\frac{r}{1+h_j},\theta\right) + \tilde{u}(r,\theta+\beta_j) +$$

$$+\tilde{u}(r,\theta-\beta_j))/4.$$

On γ_h introduce the interpolation operator I (5) $Iu \equiv (u_1\delta + u_0)/(1+\delta)$, where u_1 is the value of u at the point $P_1 \in G_h \cup \gamma$; δ is the ratio of the distance from the given node $P \in \gamma_h$ to the point P_0, lying at the intersection of γ and the straight line passing through P_1 and P, to the length of the segment $\overline{P_1P}$; u_0 is the prescribed value of u at the point P_0. It is assumed that $P \in \overline{P_1P_0} \subset G$, $\delta \le 2$, and the length of $\overline{P_1P_0}$ does not exceed $3h$. On γ_{jh} introduce the interpolation operator I_j, $I_j\tilde{u}(r,\theta) \equiv (\tilde{u}_1\theta_2 + \tilde{u}_2\theta_1)/(\theta_1+\theta_2)$, where $\tilde{u}_1$ ($\tilde{u}_2$) is the value of $\tilde{u}$ at the point $M_1(r,\theta-\theta_1)$ ($M_2(r,\theta+\theta_2)$), which either belongs to $g_{jh} \cup \gamma_{jh}$, in which case $\theta_1 = \beta_j$ ($\theta_2 = \beta_j$), or M_1 (M_2) is the point of intersection of γ and the circle of radius r, in which case $0 < \theta_1 \le \beta_j$ ($0 < \theta_2 \le \beta_j$).

On $G_{jh} \cup g^0_{jh}$ introduce the gluing operator S_j with the following properties. S_{ju} at a point $P \in G_{jh} \cup g^0_{jh}$ is expressed linearly, with nonnegative coefficients, through the values of u at points $P_k \in G_h \cup \gamma_h \cup G_{jh} \cup g_{jh} \cup g^0_{jh} \cup \gamma$, $k = 1, 2, \dots, 6$, located in the neighborhood ω_P of the point P of radius ch, c being some constant. Moreover, if $P \in G_{jh}$ ($P \in g^0_{jh}$), then the sum of the coefficients at the values of u at points belonging to $g_{jh} \cup \gamma_h \cup \gamma$ ($G_h \cup \gamma_h \cup G_{jh} \cup g_{jh} \cup \gamma$) exceeds some fixed positive constant. In addition, if u is harmonic on $\bar{\omega}_P$, then for $\omega_P = \omega_P \cap G$ one has $|u - S_{ju}| \le c^3h^3U_3$, and for $\omega_P \ne \omega_P \cap G$, $|u - S_{ju}| \le c^2h^2U_2$, where U_m is an upper bound for the modulus of the m-th derivatives of u on ω_P. An operator S_j with the indicated properties exists and can be found by the method of undetermined coefficients.

Consider the system of difference equations

$$\begin{aligned} &u_h = Au_h \quad \text{on } G_h, && u_h = Iu_h \quad \text{on } \gamma_h, \\ &u_h = A_{ju}h \quad \text{on } g_{jh}, && u_h = I_{ju}h \quad \text{on } \gamma_{jh}, \\ &u_h = S_{ju}h \quad \text{on } G_{jh} \cup g^0_{jh} \quad (j = 1, 2, \dots, N). && \end{aligned} \tag{2}$$

The system (2) has a unique solution, which can be computed by the method of iterations.

Theorem. On

$$\bigcup_{j=1}^{N} (g_{jh} \cup g^0_{jh} \cup G_{jh} \cup \gamma_{jh}) \cup G_h \cup \gamma_h$$

the inequality holds

$$|u_h - u| \leq c_0 h^2 (1 + |\ln h|), \tag{3}$$

where u_h is the solution of system (2); u is the bounded solution of problem (1); c_0 is a constant independent of h.

3. If $0 < \alpha_j < 1/2$, $\sigma_j = 1$, and the boundary values are continuous in Q_j, then one need not construct a polar grid on g_j, but may use a uniform square grid and construct the corresponding difference equations with the aid of the operators A and I. In this case an inequality of the form (3) remains valid.
4. As the basis "linking" the polar grids on g_j, $j = 1, 2, \dots, N$, one may use not a square grid, but one of the extended polar grids.
5. In the case of the exterior Dirichlet problem, one of the polar grids should be extended to a circle of radius of order h^{-2}, setting the unknowns at the nodes of the outer circle equal to the values of the unknowns on the same rays on the preceding circle. The error of the approximate solution of the exterior Dirichlet problem is of order $h^2 \ln^2 h$. The grid method for the exterior Dirichlet problem in a domain with a smooth boundary is presented in (8).
6. The composite-grid method under consideration is applicable to a multiply connected domain and, with minor modifications, generalizes to a broader class of infinite domains, for example, those whose boundary has infinite branches asymptotically equal to certain algebraic curves.

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CITED LITERATURE

1. N. S. Bakhvalov, DAN, 114, No. 3 (1957).

2. V. V. Fufaev, DAN, 131, No. 1 (1960).

3. V. A. Kondrat' ev, DAN, 153, No. 1 (1963).

4. E. A. Volkov, Tr. Mat. inst. im. V. A. Steklova AN SSSR, 77, 113 (1965).

5. L. Collatz, Zs. angew. Math. u. Mech., 13, No. 1 (1933).

6. P. Laasonen, Suomalais. Tiedeakat. Toimituks., Sar. A 1, No. 246 (1957).

7. E. A. Volkov, DAN, 155, No. 4 (1964).

8. E. A. Volkov, Zhurn. vychislit. matem. i matem. fiz., 6, No. 3 (1966).

Note: Figure translations are in progress. See original paper for figures.

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