KINETIC EQUATIONS FOR SUPERFLUID BOSE SYSTEMS AND THEIR SOLUTIONS IN THE HYDRODYNAMIC APPROXIMATION

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KINETIC EQUATIONS FOR SUPERFLUID BOSE SYSTEMS AND THEIR SOLUTIONS IN THE HYDRODYNAMIC APPROXIMATION

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In the works of N. N. Bogolyubov ([1]) and Z. Galiasevich ([2]), asymptotic expressions for the Green functions for superfluid Bose systems were obtained with the aid of the equations of hydrodynamics. However, the coefficients and poles in these expressions for the Green functions remain uncalculated. The present work is devoted to the study of kinetic equations and their solutions in the hydrodynamic approximation for weakly interacting and weakly excited Bose systems. In the work, asymptotic expressions for the Green functions are found with already known coefficients and poles.

Let us take the Hamiltonian of the system in the form

$$\mathcal{H}(\tau) = H + \delta H_\tau,$$

where δH_τ corresponds to the introduction of external sources according to ([1]); H is the Hamiltonian of a system of Bose particles with pair interaction. On the basis of our work ([3]), it is convenient to express H in the representation of Bogolyubov quasiparticles as

$$H = U + \sum_{f_1}(E(f_1) + \Delta(f_1))\xi^+_{f_1}\xi_{f_1} + \varepsilon\sum_{f_1}\frac{1}{2}S_1(f_1)(\xi^+_{f_1}\xi^+_{-f_1} + \xi_{-f_1}\xi_{f_1}) +$$

$$+ n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2}\sum_{f_1' f_2' f_1} Q(f_1', f_2'; f_1)(\xi^+_{f_1'}\xi^+_{f_2'}\xi_{f_1} + \xi^+_{f_1}\xi_{f_2'}\xi_{f_1'})\Delta(f_1' + f_2' - f_1) + \dots, \quad (1)$$

where ε is a dimensionless small parameter ([4]), and $\xi^+_{f_1}, \xi_{f_1}$ are the creation and annihilation operators of quasiparticles. The remaining notation is given in ([3]).

We define the two-time temperature Green functions for quasiparticles in the usual way:

$$\langle\langle \xi_k(t); \xi_k^+(\tau) \rangle\rangle = -i\theta(t-\tau)\langle[\xi_k(t), \xi_k^+(\tau)]_-\rangle;\ \dots \tag{2}$$

As is known, for these Green functions one can write a system of coupled equations of motion; for its approximate solution, approximate decouplings are necessary. In our case these decouplings have the form

$$\langle\langle \xi_{f_1'}^+(t)\xi_{f_2'}^+(t)\xi_{f_2}(t); \xi_k^+(\tau) \rangle\rangle = \gamma_{f_1';f_2'} n_{f_1} \Delta(f_2 - f_2')\langle\langle \xi_{f_1'}^+(t); \xi_k^+(\tau) \rangle\rangle +$$

$$+ n_0^{1/2}(\varepsilon/V)^{1/2} c_1(f_1', f_2'; f_2);$$

$$\langle\langle \xi_{f_1}^+(t)\xi_{f_2'}^-(t)\xi_{f_2}^+(t); \xi_k^+(\tau) \rangle\rangle = n_0^{1/2}(\varepsilon/V)^{1/2} c_2(f_1, f_2', f_2); \tag{3}$$

$$\langle\langle \xi_{f_1'}^+(t)\xi_{f_2'}^+(t)\xi_{f_2}(t)\xi_{f_1}(t); \xi_k^+(\tau) \rangle\rangle = \gamma_{f_1';f_2'}\gamma_{f_1;f_2} n_{f_1} \Delta(f_1 - f_1')\langle\langle \xi_{f_2'}^+(t)\xi_{f_2}(t); \xi_k^+(\tau) \rangle\rangle;$$

$$\langle\langle \xi_{f_1'}^+(t)\xi_{f_2'}^+(t)\xi_{f_2}^+(t)\xi_{f_1}(t); \xi_k^+(\tau) \rangle\rangle =$$

$$= \frac{1}{2}\gamma_{f_1';f_2';f_2} n_{f_2} \Delta(f_1 - f_1')\langle\langle \xi_{f_2'}^+(t)\xi_{f_2}(t); \xi_k^+(\tau) \rangle\rangle,$$

where

$$n_f = \left(e^{E(f)/\theta} - 1\right)^{-1}, \qquad \gamma_{f_1;\dots;f_s} = \sum_{(p)} (+1)^p$$

is the sum symmetrized over the indices. On the basis of this it is not difficult to obtain···

a closed system of equations of motion for the Green functions $\langle\langle \xi_k(t); \xi_k^+(\tau) \rangle\rangle$, $\langle\langle \xi_k^+(t); \xi_k^+(\tau) \rangle\rangle, \dots, \langle\langle \xi_{f-k/2}(t)\xi_{f+k/2}^+(t); \xi_k^+(\tau) \rangle\rangle$. Since these equations are very cumbersome, we shall not give them here. Further, in deriving the kinetic equation we restricted ourselves to terms no higher than first order in the parameter ε. In addition, all terms corresponding to multiple scatterings were discarded, and only diagonal matrix elements of three-quasiparticle processes were retained, in the sense of the work [3]. Then in the hydrodynamic approximation ($E, \mathbf{k}$ small quantities) we have the simplest form of the equations of motion for the Fourier transforms of the Green functions

$$(E - E(k))\langle\langle \xi_k \mid \xi_k^+ \rangle\rangle_E = \frac{1}{2\pi} +$$

$$+2n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2} \sum_f Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) n_f(1 + n_f) g(f, k \mid E), \tag{4}$$

$$(E + E(k))\langle\langle \xi_{-k}^+ \mid \xi_k^+ \rangle\rangle_E =$$

$$= -2n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2} \sum_f Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) n_f(1 + n_f) g(f, k \mid E),$$

where

$$\langle\langle A_k \mid B_k \rangle\rangle_E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle\langle A_k(t); B_k(\tau) \rangle\rangle e^{iE(t-\tau)}\, d(t - \tau).$$

The function

$$g(f, k \mid E) = n_f^{-1}(1 + n_f)^{-1}\langle\langle \xi_{f-k/2}^+ \xi_{f+k/2} \mid \xi_k^+ \rangle\rangle_E$$

satisfies the kinetic equation

$$-\mathcal{D}(f, k \mid g) = \mathcal{L}_s(n_f \mid g), \tag{5}$$

where

$$\mathcal{D}(f, k \mid g) = n_f(1 + n_f)\{(-E + \mathbf{k} \cdot \mathbf{v}(f)) g(f, k \mid E) +$$

$$+2n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2} \mathbf{k} \cdot \mathbf{v}(f)\, Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) \langle\langle \xi_k + \xi_{-k}^+ \mid \xi_k^+ \rangle\rangle_E, \tag{6}$$

$$\mathcal{L}_s(n_f \mid g) = \frac{8\pi}{i} n_0 \left(\frac{\varepsilon}{V}\right) \sum_{f_1', f_2'} \{|Q(f_1', f_2'; f)|^2 (1 + n_{f_1'})(1 + n_{f_2'}) n_f \times$$

$$\times \Delta(f_1' + f_2' - f)\delta(E(f_1') + E(f_2') - E(f)) \times (g(f, k \mid E) - g(f_1', k \mid E) -$$

$$-g(f_2', k \mid E)) - 2|Q(f, f_1'; f_2')|^2 (1 + n_f)(1 + n_{f_1'}) n_{f_2'} \Delta(f + f_1' - f_2') \times$$

$$\times\delta(E(f)+E(f_1')-E(f_2'))\times(g(f_2',k\mid E)-g(f_1',k\mid E)-g(f,k\mid E))\}, \quad (7)$$

where $\mathcal{L}(t)=\vec{\nabla}_f E(f)$. Obviously, $\mathcal{L}_s(n_f\mid g)$ is the collision integral, and $\mathcal{D}(f,k\mid g)$ the drift terms of the kinetic equation.

Let us proceed to discuss the solutions of the kinetic equation (5) in the hydrodynamic approximation by means of the Chapman–Enskog method [5]. We note that for $\mathcal{L}_s(n_f\mid g)$ the relations

$$0=\int f_\alpha \mathcal{L}_s(n_f\mid g)\,d^3f=\int E(f)\mathcal{L}_s(n_f\mid g)\,d^3f.$$

hold. They correspond to the laws of conservation of quasiparticle momentum and energy in collisions. Taking these conservation laws into account, we may rewrite (5) in the form of "transport equations" :

$$-Eb_1U_f^*(k\mid E)+\sum_\alpha k_\alpha \hat{P}_{j\alpha}(k\mid E)=$$

$$=-2n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2}\sum_\alpha k_\alpha A_1\delta_{j\alpha}k^{1/2}\langle\langle \xi_k+\xi_{-k}^+\mid \xi_k^+\rangle\rangle_E, \quad (8)$$

$$-E\hat{\mathcal{E}}(k\mid E)+\sum_\alpha k_\alpha \hat{Q}_\alpha(k\mid E)=0,$$

in which the following notation has been used:

$$\theta b_1 U_j^*(k|E)=\int f_j n_f(1+n_f)g(f,k|E)\,d^3f;$$

$$\hat{P}_{j\alpha}(k|E)=\int f_j v_\alpha(f)n_f(1+n_f)g(f,k|E)\,d^3f; \quad (9)$$

$$\hat{\mathcal{E}}(k|E)=\int E(f)n_f(1+n_f)g(f,k|E)\,d^3f;$$

$$\hat{Q}_\alpha(k|E)=\int v_\alpha(f)E(f)n_f(1+n_f)g(f,k|E)\,d^3f,$$

where

$$b_1=\frac{1}{3}\theta^{-2}\int f^2 n_f(1+n_f)\,d^3f;$$

$$A_1 = \int f_j v_j(t) k^{-1/2} Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) n_f(1+n_f)\, d^3 f.$$

As in the work of B. I. Sadovnikov ([6]), we put $g(f,k|E) = g^{(0)}(f,k|E) + g^{(1)}(f,k|E) + \ldots$ and require that the equalities

$$0 = \int f_\alpha n_f(1+n_f) g^{(1)}(f,k|E)\, d^3 f = \int E(f) n_f(1+n_f) g^{(1)}(f,k|E)\, d^3 f,$$

hold, i.e., that the quantities $U_j^*(k|E)$, $\hat{\mathcal{E}}(k|E)$ introduced by us be determined only through $g^{(0)}(f,k|E)$.

From expression (7) it is easy to see that $0 = \mathcal{H}_s(n_f|f_\alpha) = \mathcal{H}_s(n_f|E(f))$. Therefore, in the "acoustic approximation" ([1]) we may take $g^{(0)}(f,k|E)$ in the form

$$g^{(0)}(f,k|E) = \theta^{-1} \sum_\alpha f_\alpha U_\alpha^*(k|E) + \theta^{-2} E(f) \theta^*(k|E), \tag{10}$$

since the form $g^{(0)}$ adopted by us satisfies the equation $0 = \mathcal{H}_s(n_f|g^{(0)})$.

In the zeroth approximation, using expression (10) and definitions (9), we obtain

$$\hat{\mathcal{E}}(k|E) = b_3 \theta^*(k|E); \quad \hat{Q}_\alpha^{(0)}(k|E) = \theta b_2 U_\alpha^*(k|E); \quad \hat{P}_{j\alpha}^{(0)}(k|E) = \delta_{j\alpha} b_2 \theta^*(k|E), \tag{11}$$

where

$$b_2 = \frac{1}{3}\theta^{-2} \int \mathbf{f}\cdot\mathbf{v}(f) E(f) n_f(1+n_f)\, d^3 f, \qquad b_3 = \theta^{-2} \int E^2(f) n_f(1+n_f)\, d^3 f.$$

Substituting (11) into the transport equations (8), we find

$$\theta^*(k|E) = 2n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2}\left(\frac{b_2 A_1}{b_1 b_3}\right)\frac{k^{5/2}}{E^2 - \lambda^2 k^2}\langle\langle \xi_k + \xi_{-k}^+ \mid \xi_k^+ \rangle\rangle_E;$$

$$\sum_\alpha k_\alpha U_\alpha^*(k|E) = 2n_0^{1/2}\left(\frac{\varepsilon}{V}\right)^{1/2}\frac{k^{5/2}}{E^2 - \lambda^2 k^2} E \langle\langle \xi_k + \xi_{-k}^+ \mid \xi_k \rangle\rangle_E, \tag{12}$$

where $\lambda^2 = (b_2^2/b_1 b_3)$. Thus, we have already found $g^{(0)}(f,k|E)$, which is expressed in terms of the Green functions $\langle\langle \xi_k | \xi_k^+ \rangle\rangle_E$, $\langle\langle \xi_k^+ | \xi_k^+ \rangle\rangle_E$. Substituting this result into equation (4), we obtain

$$\langle\langle a_k \mid a_k^+ \rangle\rangle_E = -\frac{1}{2\pi} \frac{\Delta(k,E) n_0}{(E^2 - c_0^2 k^2)(E^2 - c_1^2 k^2)} = -\langle\langle a_{-k}^+ \mid a_k^+ \rangle\rangle_E, \qquad (13)$$

where

$$\lambda_1^2 = (b_3 A_1 A_2 / 2\pi^3 b_1 b_2); \qquad A_2 = \theta^{-1} \int E(f) k^{-1/2} Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right)$$

$$\times n_f (1+n_f)\, d^3 f; \qquad c_{0,1}^2 = \frac{1}{2k^2} \left\{ (E^2(k) + \lambda^2 k^2) \pm \left[(E^2(k) + \lambda^2 k^2)^2 - 4E^2(k) \lambda^2 k^2 + 8\varepsilon n_0 \lambda_1^2 E(k) k^3 \right]^{1/2} \right\}.$$

In the low-temperature region $\Delta(k,E)$ has the form $\Delta(k,E) = \nu(0)\{-E^2 + \lambda^2 k^2 + \varepsilon n_0 \lambda_1^2 (n_0 \nu(0)'/m)^{-1/2} k^2\}$. In this case all excitations are of the phonon type, and it is easy to estimate that $c_1^2 = \frac{1}{3} n_0 \nu(0)'/m = \frac{1}{3} c_0^2$—this is the well-known relation between ordinary and second sound near zero temperature.

In the first approximation the equation for $g^{(1)}(f, k|E)$ has the form $-\mathcal{D}(f, k|g^{(0)}) = \mathcal{L}_s(n_f|g^{(1)})$. Following the usual procedure, we express $\mathcal{D}(f, k|g^{(0)})$ with the aid of the transport equations of the zeroth approximation and obtain the linearized integral equation for $g^{(1)}$

$$\begin{aligned} n_f(1+n_f)\Bigg\{ &\theta^{-2} \sum_\alpha \left(v_\alpha(f) E(f) - \frac{b_2}{b_1} f_\alpha \right) k_\alpha \theta^*(k|E) + \\ &+ \theta^{-1} \sum_{\alpha,\beta} \left(f_\alpha v_\beta(f) - \frac{b_2}{b_3} E(f) \delta_{\alpha\beta} \right) k_\beta U_\alpha^*(k|E) + \\ &+ 2n_0^{1/2} \left(\frac{\varepsilon}{V}\right)^{1/2} \sum_\alpha \left(k^{-1/2} Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) - \theta^{-2} \frac{A_1}{b_1} f_\alpha \right) \times \\ &\times k^{1/2} k_\alpha \langle\langle \xi_k + \xi_{-k}^+ \mid \xi_k^+ \rangle\rangle_E \Bigg\} = \mathcal{L}_s(n_f|g^{(1)}). \end{aligned} \qquad (14)$$

The solution of equation (14) may be represented in the form

$$\begin{aligned} g^{(1)}(f,k|E) = i\Bigg\{ &\theta^{-2} \sum_\alpha \left(E(f) v_\alpha(f) - \frac{b_2}{b_1} f_\alpha \right) B_1 k_\alpha \theta^*(k|E) + \\ &+ \theta^{-1} \sum_{\alpha,\beta} \left(f_\alpha v_\beta(f) - \frac{b_2}{b_3} E(f) \delta_{\alpha\beta} \right) B_2 k_\beta U_\alpha^*(k|E) + 2n_0^{1/2} \left(\frac{\varepsilon}{V}\right)^{1/2} \times \\ &\times \sum_\alpha \left(k^{-1/2} Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) - \theta^{-2} \frac{A_1}{b_1} f_\alpha \right) B_3 k^{1/2} k_\alpha \langle\langle \xi_k + \xi_{-k}^+ \mid \xi_k^+ \rangle\rangle_E \Bigg\}. \end{aligned} \qquad (15)$$

For simplicity we regard B_1, B_2, B_3 as constants. Then $g^{(1)}(f,k|E)$ automatically satisfies the condition $0 = \int f_\alpha n_f(1+n_f) g^{(1)}(f,k|E)\, d^3f = \int E(f) n_f(1+n_f) g^{(1)}(f,k|E)\, d^3f$, and it is easy to find the quantities B_1, B_2, B_3 from equation (14). Using the expressions obtained in (10), (15) and the transport equations (8), by a method analogous to that used in the zeroth approximation, we obtain the one-particle Green function

$$\langle\langle a_k \mid a_k^+ \rangle\rangle_{E+i\varepsilon} \simeq -\frac{\Delta(k,E) n_0}{2\pi c_0 c_1 k^2 (E + 2i\varepsilon_0 - c_0 k)(E + 2i\varepsilon_1 - c_1 k)}, \tag{16}$$

where

$$\varepsilon_0 = c_0^{-2}\left\{\frac{1}{2}(E^2(k) - c_0^2 k^2)\Gamma_1 + \varepsilon n_0 \lambda_1^2 k E(k) \Gamma_2\right\}; \quad \varepsilon_1 = c_0^{-2}\left\{\frac{1}{2}E^2(k)\Gamma_1 + \varepsilon n_0 E_1 k E(k)\Gamma_2\right\};$$

$$\Gamma_1 = \left(\frac{b_2^2}{b_1 b_3} - \frac{b_4}{b_3}\right) B_1 + 2\left(\frac{b_2^2}{b_1 b_3} - \frac{b_5}{b_1}\right)\frac{B_2}{\theta}; \quad \Gamma_2 = \left(\frac{b_1 A_3}{b_2 A_1} - 1\right)\theta B_2;$$

$$b_4 = \frac{1}{3}\theta^{-2}\int v^2(f) E^2(f) n_f(1+n_f)\, d^3f; \quad b_5 = \frac{1}{q}\theta^{-2}\int f^2 v^2(f) n_f(1+n_f)\, d^3f;$$

$$A_3 = \frac{1}{3}\theta^{-1}\int v^2(f) E(f) k^{-1/2} Q\left(k, f - \frac{k}{2}; f + \frac{k}{2}\right) n_f(1+n_f)\, d^3f.$$

Comparing (16) with the result of Galyasevich[2], we see that now the absorption coefficients of first and second sound, $\varepsilon_0, \varepsilon_1$, are already expressed in terms of known quantities related to the properties of the substance.

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Note: Figure translations are in progress. See original paper for figures.