EMBEDDING OF TREE-LIKE COMPACTA IN \(E^3\)

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Abstract

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MATHEMATICS

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EMBEDDING OF TREE-LIKE COMPACTA IN E^3

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1. A metric compactum which, for every $\varepsilon > 0$, has an ε-covering whose nerve is a finite number of trees is called **tree-like**. In this paper an outline is given of the proof that the class of tree-like compacta coincides with the class of those one-dimensional compacta that can be embedded cellularly in three-dimensional Euclidean space. In addition, a simple criterion is given that characterizes cellularly decomposed compacta in E^3 by means of properties of the complementary space.

2. Definition 1. A compactum $K \subset E^n$ is called **cellularly decomposed** in E^n if it is the intersection of a countable number of finite sums of topological cubes:

$$K = \bigcap_{n=1}^{\infty} L_n,$$

where $L_{n+1} \subset \operatorname{int} L_n$ $(n = 1, 2, ...)$, and L_n is a sum of a finite number of nonintersecting topological cubes

$$L_n = L_n^1 \cup L_n^2 \cup ... \cup L_n^{s_n}.$$

Definition 2. A compactum $K \subset E^n$ is called **uniformly cellularly decomposed** in E^n if it can be represented as the intersection of a countable number of finite sums of topological cubes

$$K = \bigcap_{n=1}^{\infty} L_n$$

such that $L_{n+1} \subset \operatorname{int} L_n$ $(n = 1, 2, ...)$, and

$$L_n = L_n^1 \cup L_n^2 \cup ... \cup L_n^{s_n}, \tag{1}$$

each L_n^i is a topological cube whose diameter is less than ε_n, where $\varepsilon_n > 0$ $(n = 1, 2, ...)$ and $\varepsilon_n \to 0$ as $n \to \infty$; no three cubes in (1) have common points; any two cubes either do not intersect, or intersect in a disk and have no common interior points; the nerve of the system $(L_n^1; L_n^2; \dots ; L_n^{s_n})$ is a sum of a finite number of nonintersecting trees.

Definition 3. The closure of a bounded domain of the space E^n is called a **simple closed domain** if its boundary is a manifold.

Theorem 1. For every tree-like compactum K in E^3 $(K \subset E^3)$ and for every $\varepsilon > 0$ there exists a homeomorphic ε-mapping of the compactum K onto a uniformly cellularly decomposed compactum K_ω in E^3 $(h_\varepsilon : K \xrightarrow{\text{onto}} K_\omega)$.

The proof is based on the following lemmas.

Lemma 1. For every tree-like compactum $K \subset E^3$ and for every $\varepsilon > 0$ there exists a finite number of simple closed domains of the space E^3: $Q_1, Q_2, \dots , Q_s$, such that:

1) $\operatorname{diam}(Q_i) < \varepsilon$ $(i = 1, 2, \dots , s)$;
2) $Q_i \cap Q_j \cap Q_k = \Lambda$, if all indices are distinct;
3) $\operatorname{int}(Q_i) \cap \operatorname{int}(Q_j) = \Lambda$, if $i \neq j$;
4) $Q_i \cap Q_j$ is either empty or is a finite sum of nonintersecting topological squares;
5) $\operatorname{int}\left(\bigcup_i Q_i\right) \supset K$;
b) the nerve of the system $(Q_1, Q_2, \dots , Q_s)$ is the sum of a finite number of nonintersecting trees.

Lemma 2. Let

$$U = \operatorname{int}\left(\bigcup_{i=1}^{s} Q_i\right)$$

be a neighborhood of a dendroid compactum K satisfying the condition of Lemma 1; then there exists a simple closed region V such that $U \supset V$, and V is homeomorphic to the sum of a finite number of topological cubes, with $V \cap Q_i$ a topological cube; moreover, there exists an ε-homeomorphism mapping the compactum K into $\operatorname{int} V$ (the number ε satisfies the conditions of Lemma 1).

Lemma 3. Let $K_1, K_2, \dots , K_n, \dots$ be a sequence of compacta lying in a bounded part of Euclidean space E^n, and suppose that for every n there exists a homeomorphic mapping

$$h_n : K_n \xrightarrow{\text{onto}} K_{n+1}$$

$(n = 1, 2, \dots)$, which is an ε_n-shift. If the sequence of positive numbers $\varepsilon_n > 0$ for every n satisfies the condition

$$\sum_{k=n+1}^{\infty} \varepsilon_k < \frac{1}{2} \min \rho(h_n \cdots h_1(x);\ h_n \cdots h_1(y))$$

over all $x, y \in K_1$ such that $\rho(x,y) \geq \delta_n$, where $\delta_n \to 0$ as $n \to \infty$, then the sequence of compacta $K_1, K_2, \dots, K_n, \dots$ converges to the compactum

$$K_\omega = \mathrm{lt}_{n\to\infty} K_n,$$

and K_1 is homeomorphic to the compactum K_ω, and this homeomorphism is an ε-shift,

$$\varepsilon = \sum_{n=1}^{\infty} \varepsilon_n.$$

For the proof of Theorem 1, Lemmas 1, 2, 3 are applied successively. Lemma 3 can be applied, since on the basis of Lemma 2 we can homeomorphically ε_n-shift the compactum K_n into the corresponding neighborhood V_n, for arbitrary $\varepsilon_n \to 0$ $(n \to \infty)$ and independently for different n.

If in the condition of Lemma 3 we take

$$\sum_{k=1}^{\infty} \varepsilon_n = \varepsilon,$$

then in the limit, as $n \to \infty$, we obtain a compactum K_ω, homeomorphic to the compactum $K = K_1$, the homeomorphism being an ε-shift $(h_\varepsilon : K_1 \to K_\omega)$.

The compactum K_ω is the intersection of simple closed neighborhoods

$$V_n : K_\omega = \bigcap_{n=1}^{\infty} V_n,$$

satisfying the conditions of Definition 2.

3. Definition 4. For a compactum $K \subset E^n$, the number $d_2(K)$ is the exact lower bound of all numbers $\varepsilon > 0$ such that there exists a finite closed ε-covering of the compactum K whose multiplicity is not greater than 2.

Definition 5. For a compactum $K \subset E^n$, the number $A_2(K)$ is the exact lower bound of all numbers $\varepsilon > 0$ such that there exists a finite closed ε-covering of the compactum K, every component of whose nerve is a tree.

Theorem 2. Every one-dimensional cellularly decomposable compactum in E^n, $K \subset E^n$, is dendroid.

For the proof the following lemma is needed.

Lemma 4. If $L^n \subset E^n$ is a topological cube, then

$$d_2(L^n) = A_2(L^n).$$

It follows from Theorems 1 and 2 that, in order that a one-dimensional compactum K can be cellularly decomposably embedded in E^3, it is necessary and sufficient that K be dendroid.

4. Definition 6. A compactum $K \subset E^n$ is called **spherically embedded** if for every compactum $C \subset E^n \setminus K$ that does not separate ⋯

E^n, there exists a topological sphere S^{n-1} such that $\operatorname{int}(S^{n-1}) \supset K$, $\operatorname{ext}(S^{n-1}) \supset C$, $S^{n-1} \cap (K \cup C) = \Lambda$.

In the case $n = 3$, by Bing' s approximation theorem (^{1}), the sphere S^2 may be assumed polyhedral.

Theorem 3. *In order that a compactum $K \subset E^3$ be cellularly separated, it is necessary and sufficient that it be spherically embedded.*

We outline the idea of the proof of necessity. If U is an arbitrary neighborhood of the compactum K, then there exists a finite number of simply connected closed domains $Q_1, Q_2, \dots, Q_s$ such that

$$U \supset \bigcup_{i=1}^{s} Q_i \supset \bigcap_{i=1}^{s} \operatorname{int}(Q_i) \supset K.$$

Consider one of the domains Q_i, denoting it by Q. From the boundary of the domain Q remove a disk D; then $M^2 = \operatorname{fr}(Q) \setminus D$ does not split E^3, and, by the hypothesis, there exists a sphere S^2 such that $\operatorname{int}(S^2) \supset K$, $\operatorname{ext}(S^2) \supset M^2$, $S^2 \cap (K \cup M^2) = \Lambda$. Moreover, it may be assumed that $S^2 \cap D$ consists of a finite number of disjoint circles. To prove the theorem we replace the domain Q by a finite number of disjoint topological cubes covering the compactum $\operatorname{int}(Q) \cap K$. This can be done by cutting the cube $S^2 \cup \operatorname{int}(S^2)$ "into parts," using the properties of the intersection $S^2 \cap D$ and observing that, by Alexander' s theorem, $S^2 \cup \operatorname{int}(S^2)$ is a topological cube.

Theorem 4. *Every subcompactum of a one-dimensional cellularly separated compactum in E^3 is also cellularly separated.*

In the proof one applies Theorem 2 and the idea of the proof of Theorem 3.

Question 1. Is Theorem 4 true for E^n when $n > 3$? McMillan gives an affirmative answer to this question for embeddings of cellular arcs in E^n $(n \neq 4)$ (^{2}).

The following theorem strengthens the result of Theorem 3.

Theorem 5. *In order that a compactum $K \subset E^3$ be cellularly separated, it is necessary and sufficient that it be spherically separable from every one-dimensional polyhedron lying in the complement.*

The proof rests on a technical lemma asserting that a two-dimensional polyhedral manifold with boundary lying in E^3 has a complement homeomorphic to the complement of some one-dimensional polyhedron.

Question 2. Is it true that a compactum $K \subset E^3$ is cellularly separated if and only if it is spherically separable from every polyhedral neighborhood in the complement?

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Note: Figure translations are in progress. See original paper for figures.

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