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DISCRETE PERIODIC PROCESSES

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Abstract

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MATHEMATICS

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DISCRETE PERIODIC PROCESSES

(Presented by Academician I. M. Vinogradov on 26 XI 1965)

Let an infinite increasing sequence of moments of time be given,

$$\tau_1, \tau_2, \dots, \tau_x, \dots \tag{1}$$

An example of a discrete process occurring at the moments of time τ_x and periodic in the index x is provided by the following problem.

Problem 1. Materials enter a bunker through s channels, and materials are issued from the bunker through other t channels. The channels for the arrival of materials are numbered $1, \dots, s$, and the expenditure channels are numbered from 1 to t. The arrival of materials into the bunker and the issue from it can occur only at moments of time that are members of the sequence (1). Loading into and unloading from the bunker take place according to a plan Π, which consists of the following. The arrival of materials through the i-th channel, $i = 1, \dots, s$, begins at a definite moment with number m_i and then is repeated periodically with period a_i; each time A_i units of material enter the bunker through the i-th channel. The issue of material through the j-th channel, $j = 1, \dots, t$, begins at the moment of time with number n_j and then is repeated periodically with period b_j; each time B_j units of material are issued from the bunker through the j-th channel. The condition is imposed that the bunker be merely a distribution point—there must not occur in it an unbounded accumulation of material, nor must an unboundedly growing deficit form. In other words, the average arrival of material must be equal to its average expenditure. This condition is expressed by the formula

$$\sum_{i=1}^{s} \frac{A_i}{a_i} = \sum_{j=1}^{t} \frac{B_j}{b_j}. \tag{2}$$

However, an arbitrarily prescribed plan Π may turn out to be infeasible because, after a certain number of moments of time from the sequence (1), the total arrival of material may prove to be less than the amount of material that must be issued from the bunker. In view of this, in order to ensure uninterrupted operation of the bunker, one should from the very beginning, or at the necessary moment, give the bunker an additional quantity of material. This additional quantity of material must, naturally, be minimal; it is this quantity that is required to be determined.

Passing to the solution of the problem posed, for simplicity let us suppose that the conditions

$$1 \leq m_i \leq a_i \quad (i = 1, \dots, s), \qquad 1 \leq n_j \leq b_j \quad (j = 1, \dots, t). \tag{3}$$

are satisfied. Let x denote the number of a moment of time. By $r(x)$ we shall denote the remainder of material in the bunker after the first x moments of its operation according to plan Π (therefore, $r(x)$ may have negative val-

tions). Taking into account conditions (3), it is easy to obtain, for $x = 1, 2, \dots$, the formula

$$r(x) = \sum_{i=1}^{s} A_i \left[\frac{x + a_i - m_i}{a_i}\right] - \sum_{j=1}^{t} B_j \left[\frac{x + b_j - n_j}{b_j}\right],$$

in which the square brackets denote the integer part of the number enclosed in these brackets. From condition (2) it follows that $r(x)$ has period $k = \mathrm{l.\,c.\,m.}[a_1, \dots, a_s, b_1, \dots, b_t]$. It is clear that the required additional quantity of material is equal to

$$\left|\min\left\{0, \min_x r(x)\right\}\right|.$$

Thus, the problem posed reduces to finding $\min r(x)$ for $x = 1, 2, \dots$. By elementary number-theoretic reasoning one can show that the problem of finding the extremum (maximum or minimum) of $r(x)$, in turn, reduces to the following problem.

Problem 2. Determine the numbers f_i, $i = 1, \dots, s$, h_j, $j = 1, \dots, t$, from the conditions:

1. $1 \leq f_i \leq a_i$, $1 \leq h_j \leq b_j$.
2. The system of congruences

$$\begin{aligned} x &\equiv m_i - f_i \pmod{a_i}, \quad i = 1, \dots, s, \\ x &\equiv n_j - h_j \pmod{b_j}, \quad j = 1, \dots, t, \end{aligned} \tag{4}$$

must be consistent.

3. The linear form

$$L = \sum_{i=1}^{s} \frac{A_i}{a_i} x_i - \sum_{j=1}^{t} \frac{B_j}{b_j} y_j,$$

in which the integer variables x_i, y_j are subject to the inequalities $1 \le x_i \le a_i,\ 1 \le y_j \le b_j$, must have the extremal value L_0 (maximum or minimum) at $x_i = f_i,\ y_j = h_j$.

Indeed, it is not hard to calculate that

$$\operatorname*{extr}_x r(x) = -\sum_{i=1}^{s} \frac{A_i}{a_i} m_i + \sum_{j=1}^{t} \frac{B_j}{b_j} n_j + L_0.$$

Problem 2 is of interest because it represents a peculiar problem of linear programming for congruences.

If the moduli of the system of congruences (4) are pairwise relatively prime, then it is consistent for any values of f_i, h_j (see (1), Ch. 4, § 3). In this case the minimum of L is attained for $f_i = 1,\ h_j = b_j$, and the maximum of L for $f_i = a_i,\ h_j = 1$.

If not all moduli of the system of congruences (4) are pairwise relatively prime, then, for the consistency of this system, f_i and h_j must satisfy certain conditions. These conditions can be obtained from the system of congruences (4) by using the following lemma.

Consistency lemma. Let

$$M = \text{l. c. m.}[M_1, M_2, \dots, M_r]$$

and let the canonical decompositions be known:

$$M = p_1^{\alpha_1} \cdots p_u^{\alpha_u}, \qquad M_\nu = p_1^{\beta_{1\nu}} \cdots p_u^{\beta_{u\nu}}, \qquad \nu = 1, \dots, r$$

(among the numbers $\beta_{\alpha\nu}$ there may be zeros). The system of congruences

$$x \equiv c_\nu \pmod{M_\nu}, \qquad \nu = 1, \dots, r, \tag{5}$$

simultaneously if and only if

$$c_\nu \equiv c_{\nu_\gamma} \pmod{p_\gamma^{\beta_{\gamma\nu}}}, \qquad \nu = 1, \dots, r; \quad \gamma = 1, \dots, u, \tag{6}$$

where the index ν_γ is determined, generally speaking, ambiguously, from the condition

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$$\beta_{\gamma\nu_\gamma} = \max\{\beta_{\gamma 1}, \dots, \beta_{\gamma r}\}.$$

When conditions (6) are satisfied, the system of congruences (5) is equivalent to the system of congruences

$$x \equiv c_{\nu_\gamma} \pmod{p_\gamma^{\beta_{\gamma\nu_\gamma}}}, \qquad \gamma = 1, \dots, u.$$

The proof of the lemma is simple, and is omitted here.

Problem 3. As in Problem 1, the bunker operates at the instants of time forming the sequence (1). Material arrives through s channels. Along the i-th channel, beginning with the instant of time numbered m_i, equal amounts A_i of material arrive with period a_i. There is one discharge channel, through which, beginning with the instant of time numbered n, all the material accumulated in the bunker is unloaded with period b. It is required to determine the greatest and the least amount of material that is unloaded from the bunker at one time.

This problem reduces to Problem 1. The numbers y of those instants of time at which the bunker is unloaded have the form

$$y = bz + n, \qquad z = 0, 1, \dots.$$

It is easy to see that the state of the bunker at the instants y is described by the function

$$r(z) = \sum_{i=1}^{s} A_i \left[\frac{bz + n + a_i - m_i}{a_i}\right] - \sum_{i=1}^{s} A_i \left[\frac{b(z-1) + n + a_i - m_i}{a_i}\right]$$

for $z = 1, 2, \dots$. The function $r(z)$ is studied in the same way as the function $r(x)$ in Problem 1.

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REFERENCES

1. I. M. Vinogradov, *Foundations of Number Theory*, Moscow, 1965.

Note: Figure translations are in progress. See original paper for figures.

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