ON UNIVALENT FUNCTIONS WITHOUT COMMON VALUES IN A MULTIPLY CONNECTED DOMAIN

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Abstract

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Yu. E. ALENITSYN

ON UNIVALENT FUNCTIONS WITHOUT COMMON VALUES IN A MULTIPLY CONNECTED DOMAIN

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In the present note some new results are given for univalent functions without common values in a finitely connected domain, obtained from the consideration of suitable Dirichlet integrals.

Let B be a finite and finitely connected domain of the z-plane, bounded by closed analytic Jordan curves; $\mathcal{L}^2(B)$ the class of all functions regular and with integrable squared modulus in the domain B; $\mathcal{L}_0^2(B)$ the subclass of functions from $\mathcal{L}^2(B)$ with single-valued integrals in the domain B; $K(z,\zeta)$ and $l(z,\zeta)$ the Bergman kernels ([1]) of the first and second kind of the domain B of the class $\mathcal{L}^2(B)$; $K_0(z,\zeta)$ and $l_0(z,\zeta)$ those of the class $\mathcal{L}_0^2(B)$;

$$U_k(z,\zeta) = \frac{1}{\pi}\left[\frac{f_k'(z) f_k'(\zeta)}{(f_k(z) - f_k(\zeta))^2} - \frac{1}{(z-\zeta)^2}\right], \quad z, \zeta \in B,$$

where the function $f_k(z)$ is regular* and univalent in the domain B.

Theorem 1. If the functions $f_k(z)$, $k = 1, \dots, n$, are regular, univalent, and without common values in the domain B, then for any points $\zeta_{k\mu}$ of this domain and any constants $\alpha_{k\mu}$, $\mu = 1, \dots, N$, $k = 1, \dots, n$, the inequality holds:

$$\left|\sum_{\mu,\nu=1}^{N}\left\{\sum_{k=1}^{n}\alpha_{k\mu}\alpha_{k\nu}[U_k(\zeta_{k\mu},\zeta_{k\nu}) + l_0(\zeta_{k\mu},\zeta_{k\nu})] + \frac{2}{\pi}\sum_{1\le j<k\le n}\alpha_{j\mu}\alpha_{k\nu}\frac{f_j'(\zeta_{j\mu}) f_k'(\zeta_{k\nu})}{(f_j(\zeta_{j\mu}) - f_k(\zeta_{k\nu}))^2}\right\}\right| \le \sum_{\mu,\nu=1}^{N}\sum_{k=1}^{n}\alpha_{k\mu}\overline{\alpha}_{k\nu}K_0 \tag{1^0}$$

This inequality remains valid also after the simultaneous replacement in it of the functions l_0 and K_0, respectively, by the functions l and K (this second inequality will henceforth be cited as inequality (1)).

One can indicate necessary and sufficient conditions under which equality holds in the inequalities (1^0) and (1)).

Each of the inequalities (1^0) and (1) determines the corresponding disk in which lie the possible values of the functional

$$\sum_{\mu,\nu=1}^{N}\left[\sum_{k=1}^{n}\alpha_{k\mu}\alpha_{k\nu}U_k(\zeta_{k\mu},\zeta_{k\nu})+\frac{2}{\pi}\sum_{1\le j<k\le n}\alpha_{j\mu}\alpha_{k\nu}\frac{f_j'(\zeta_{j\mu})f_k'(\zeta_{k\nu})}{(f_j(\zeta_{j\mu})-f_k(\zeta_{k\nu}))^2}\right]$$

for fixed $\zeta_{k\mu}$ and $\alpha_{k\mu}$, $\mu=1,\dots,N$; $k=1,\dots,n$.

The disk determined by inequality (1^0) lies in the disk determined by inequality (1).

For $n=1$, inequalities (1^0) and (1) give inequalities previously obtained by another method by Bergman and Schiffer ([1]).

Corollary 1. If the function $f(z)$ is regular, univalent, and bounded in the domain B: $|f(z)|<1$, $z\in B$, then for any points ζ_μ of this domain

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* Here and below—including single-valuedness.

and any constants α_μ, $\mu=1,\dots,N$, the inequality holds:

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_\mu\alpha_\nu\left[U(\xi_\mu,\xi_\nu)+l_0(\xi_\mu,\xi_\nu)\right]\right|\le\sum_{\mu,\nu=1}^{N}\alpha_\mu\bar{\alpha}_\nu\left[K_0(\xi_\mu,\bar{\xi}_\nu)-\frac{f'(\xi_\mu)\overline{f'(\xi_\nu)}}{\pi(1-f(\xi_\mu)\overline{f(\xi_\nu)})^2}\right]. \tag{2}$$

In particular:

$$\left|\frac{1}{6}\{f,z\}+\pi l_0(z,z)\right|+\frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\le\pi K_0(z,\bar{z}),\qquad z\in B,$$

where $\{f,z\}$ is the Schwarzian invariant.

Inequality (2) strengthens Singh's inequality [(2)] for bounded univalent functions, in which l_0 and K_0 are replaced by l and K.

Let $\widetilde{C}(B)$ denote the class of all functions $f(z)$, regular and univalent in the domain B, satisfying the condition: $f(z_1)f(z_2)\ne1$ for any points z_1 and z_2 of the domain B.

Corollary 2. If $f(z)\in\widetilde{C}(B)$, then for any points ξ_μ of the domain B and any constants α_μ, $\mu=1,\dots,N$, the inequality holds:

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_\mu\alpha_\nu\left[U(\xi_\mu,\xi_\nu)\pm\frac{f'(\xi_\mu)f'(\xi_\nu)}{\pi(1-f(\xi_\mu)f(\xi_\nu))^2}+l_0(\xi_\mu,\xi_\nu)\right]\right|\le$$

$$\le\sum_{\mu,\nu=1}^{N}\alpha_\mu\bar{\alpha}_\nu K_0(\xi_\mu,\bar{\xi}_\nu),$$

In particular:

$$\left|\frac{1}{6}\{f,z\} \pm \left(\frac{f'(z)}{1-f^2(z)}\right)^2 + \pi l_0(z,z)\right| \le \pi K_0(z,\bar{z}), \quad z \in B.$$

Each of these inequalities remains valid after replacing l_0 and K_0 in it respectively by l and K.

Theorem 2. Let the functions $f_k(z)$, $k = 1, \dots, n$, be regular in a domain B containing the origin and satisfy the conditions: $f_j(0) \neq f_k(0)$, $j \neq k$, $f'_k(0) \neq 0$. In order that these functions be univalent functions without common values in the domain B, it is necessary and sufficient that, for any n complex vectors $\{\alpha_{k0}, \dots, \alpha_{kN}\}$, $k = 1, \dots, n$, with arbitrary $N = 0, 1, 2, \dots$, the conditions

$$\sum_{\mu,\nu=0}^{N}\left[\sum_{k=1}^{n}\alpha_{k\mu}\alpha_{k\nu}(c_{\mu\nu}^{[k]} + \lambda_{\mu\nu}) + \sum_{1\le j<k\le n}\alpha_{j\mu}\alpha_{k\nu}d_{\mu\nu}^{[jk]}\right] \le \sum_{\mu,\nu=0}^{N}\varkappa_{\mu\nu}\sum_{k=1}^{n}\alpha_{k\mu}\bar{\alpha}_{k\nu},$$

be satisfied, where $c_{\mu\nu}^{[k]}$, $d_{\mu\nu}^{[jk]}$, $\lambda_{\mu\nu}$, and $\varkappa_{\mu\nu}$ are determined by the following expansions into double series in a neighborhood of the origin:

$$U_k(z,\zeta) = \sum_{\mu,\nu=0}^{\infty} c_{\mu\nu}^{[k]} z^\mu \zeta^\nu, \qquad k = 1, \dots, n;$$

$$\frac{2f'_j(z)f'_k(\zeta)}{\pi[f_j(z) - f_k(\zeta)]^2} = \sum_{\mu,\nu=0}^{\infty} d_{\mu\nu}^{[jk]} z^\mu \zeta^\nu, \qquad 1 \le j < k \le n;$$

$$l(z,\zeta) = \sum_{\mu,\nu=0}^{\infty}\lambda_{\mu\nu} z^\mu \zeta^\nu, \qquad K(z,\bar{\zeta}) = \sum_{\mu,\nu=0}^{\infty}\varkappa_{\mu\nu} z^\mu \bar{\zeta}^\nu.$$

For $n = 1$ this theorem gives the known [1] necessary and sufficient conditions for univalence of a function in a multiply connected domain.

Corollary. Let the function $f(z)$ be regular in the domain B, containing the origin, and satisfy the conditions $f(0) \neq \pm 1$, $f'(0) \neq 0$. In order that the function $f(z)$ belong to the class $\widetilde{C}(B)$, it is necessary and sufficient that, for any two complex vectors $\{a_{k0}, \dots, a_{kN}\}$, $k = 1, 2$, with arbitrary $N = 0, 1, 2, \dots$, the conditions

$$\left|\sum_{\mu,\nu=0}^{N}\left[(\alpha_{1\mu}\alpha_{1\nu} + \alpha_{2\mu}\alpha_{2\nu})(c_{\mu\nu} + \lambda_{\mu\nu}) - \alpha_{1\mu}\alpha_{2\nu}d_{\mu\nu}\right]\right| \le \sum_{\mu,\nu=0}^{N}(\alpha_{1\mu}\overline{\alpha}_{1\nu} + \alpha_{2\mu}\overline{\alpha}_{2\nu})\,\chi_{\mu\nu},$$

hold, where the coefficients $c_{\mu\nu}$, $\lambda_{\mu\nu}$, and $\chi_{\mu\nu}$ are defined in Theorem 2, while $d_{\mu\nu}$ are determined by the expansion in a neighborhood of the origin

$$\frac{2f'(z)f'(\zeta)}{\pi[1-f(z)f(\zeta)]^2} = \sum_{\mu,\nu=0}^{\infty} d_{\mu\nu} z^{\mu} \zeta^{\nu}.$$

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.