

ON CONSERVATION LAWS IN EINSTEIN' S THEORY OF GRAVITATION

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Abstract

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PHYSICS

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ON CONSERVATION LAWS IN EINSTEIN' S THEORY OF GRAVITATION

(Presented by Academician V. A. Fock on 11 XII 1965)

The paper considers the connection between conservation laws in Einstein' s theory of gravitation and the properties of the Lagrangian function of the gravitational field.

We shall start from the Einstein gravitational equations in Lagrangian form:

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x_\alpha} \frac{\partial \mathcal{L}}{\partial (\partial g_{\mu\nu} / \partial x_\alpha)} = -\varkappa \sqrt{-g} T^{\mu\nu}, \quad (1)$$

where

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} (\Gamma_{\mu\alpha}^\beta \Gamma_{\nu\beta}^\alpha - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta) \quad (2)$$

is the Lagrangian function of the gravitational field; $g_{\mu\nu}$ is the fundamental tensor; g is the determinant composed of the components of the fundamental tensor $g_{\mu\nu}$; $\Gamma_{\mu\nu}^\alpha$ are Christoffel symbols of the second kind; \varkappa is Einstein' s gravitational constant, and $T^{\mu\nu}$ is the mass tensor. Greek indices take the values 0, 1, 2, 3. Summation over identical Greek indices from 0 to 3 is assumed.

For the purpose of deriving conservation laws, following works ⁽¹⁻⁴⁾, let us pass from the variables x_0, x_1, x_2, x_3 to new variables x'_0, x'_1, x'_2, x'_3 , putting

$$x'_\beta = x_\beta(x_0, x_1, x_2, x_3, \varepsilon), \quad (3)$$

where ε is a certain parameter, and

$$(x'_\beta)_{\varepsilon=0} = x_\beta. \quad (4)$$

Taking into account that the Lagrangian function (2) depends only on $g_{\mu\nu}$ and their first derivatives with respect to the variables x_α , and noting that

$$\left(\frac{\partial}{\partial \varepsilon} \frac{\partial g'_{\mu\nu}}{\partial x'_\alpha}\right)_{\varepsilon=0} = \frac{\partial}{\partial x_\alpha} \left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0}, \quad (5)$$

we obtain

$$\begin{aligned} \left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon}\right)_{\varepsilon=0} &= \mathcal{L} \frac{\partial}{\partial x_\beta} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} \\ &\quad - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0}. \end{aligned} \quad (6)$$

Here

$$\mathcal{L}^* = \mathcal{L}' \left| \frac{D(x'_0, x'_1, x'_2, x'_3)}{D(x_0, x_1, x_2, x_3)} \right|, \quad (7)$$

$$\mathcal{L}' = \sqrt{-g'} g'^{\mu\nu} (\Gamma'_{\mu\alpha}{}^\beta \Gamma'_{\nu\beta}{}^\alpha - \Gamma'_{\mu\nu}{}^\alpha \Gamma'_{\alpha\beta}{}^\beta). \quad (2')$$

If we now make use of the gravitational equations (1) and the equalities

$$\left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} = -g_{\mu\cdot} \frac{\partial}{\partial x_\nu} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} - g_{\nu\cdot} \frac{\partial}{\partial x_\mu} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0}, \quad (8)$$

then we shall have

$$\begin{aligned} \left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon}\right)_{\varepsilon=0} &= \left(\mathcal{L} \delta_\beta^\alpha - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} \right) + 2\kappa \sqrt{-g} g T_\beta^\alpha \frac{\partial}{\partial x_\alpha} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} \\ &\quad - 2 \frac{\partial}{\partial x_\alpha} \left[g_{\mu\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} \frac{\partial}{\partial x_\nu} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} \right], \end{aligned} \quad (9)$$

where $\delta_\beta^\alpha = 1$ for $\alpha = \beta$ and $\delta_\beta^\alpha = 0$ for $\alpha \neq \beta$. Consequently, taking into account Einstein's conservation laws,

$$\frac{\partial}{\partial x_\alpha} \sqrt{-g} (T_\beta^\alpha + \theta_\beta^\alpha) = 0, \quad (10)$$

where

$$\theta_{\beta}^{\alpha} = \frac{1}{2\kappa\sqrt{-g}} \left(\mathcal{L}\delta_{\beta}^{\alpha} - \frac{\partial g_{\mu\nu}}{\partial x_{\beta}} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_{\alpha})} \right) \quad (11)$$

is Einstein's energy-momentum pseudotensor, we arrive at the conclusion that

$$\left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon} \right)_{\varepsilon=0} = \frac{\partial}{\partial x_{\alpha}} \left\{ 2\kappa\sqrt{-g} (T_{\beta}^{\alpha} + \theta_{\beta}^{\alpha}) \left(\frac{\partial x'_{\beta}}{\partial \varepsilon} \right)_{\varepsilon=0} - 2g_{\mu\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_{\alpha})} \frac{\partial}{\partial x_{\nu}} \left(\frac{\partial x'_{\beta}}{\partial \varepsilon} \right)_{\varepsilon=0} \right\}. \quad (12)$$

In deriving equality (12), the equations of gravitation (1) were essentially used. We shall now calculate the same quantity $(\partial \mathcal{L}^*/\partial \varepsilon)_{\varepsilon=0}$, but without taking account of the equations of gravitation. For this purpose we shall use the transformation law of the Lagrangian function (2). As is known (see (5)),

$$\mathcal{L}' \left| \frac{D(x'_0, x'_1, x'_2, x'_3)}{D(x_0, x_1, x_2, x_3)} \right| = \mathcal{L} + \frac{\partial}{\partial x_{\alpha}} (P_{\mu\nu}^{\alpha} \mathfrak{G}^{\mu\alpha} - P_{\mu\nu}^{\alpha} \mathfrak{G}^{\mu\nu}), \quad (13)$$

$$P_{\mu\nu}^{\alpha} = \frac{\partial x'_{\sigma}}{\partial x_{\mu}} \frac{\partial x'_{\tau}}{\partial x_{\nu}} \frac{\partial^2 x_{\alpha}}{\partial x'_{\sigma} \partial x'_{\tau}}, \quad \mathfrak{G}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \quad (14)$$

Since

$$\left(\frac{\partial P_{\mu\nu}^{\alpha}}{\partial \varepsilon} \right)_{\varepsilon=0} = - \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} \left(\frac{\partial x'_{\alpha}}{\partial \varepsilon} \right)_{\varepsilon=0}, \quad (15)$$

we have, according to (13),

$$\left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon} \right)_{\varepsilon=0} = - \frac{\partial}{\partial x_{\alpha}} \left\{ \frac{\partial \mathfrak{G}^{\mu\nu}}{\partial x_{\mu}} \frac{\partial}{\partial x_{\nu}} \left(\frac{\partial x'_{\alpha}}{\partial \varepsilon} \right)_{\varepsilon=0} - \frac{\partial \mathfrak{G}^{\mu\alpha}}{\partial x_{\nu}} \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial x'_{\nu}}{\partial \varepsilon} \right)_{\varepsilon=0} \right\}. \quad (16)$$

Comparing (12) and (16), we arrive at the following conservation laws

$$\frac{\partial}{\partial x_{\alpha}} \left\{ 2\kappa\sqrt{-g} (T_{\beta}^{\alpha} + \theta_{\beta}^{\alpha}) \varphi^{\beta} - 2g_{\mu\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_{\alpha})} \frac{\partial \varphi^{\beta}}{\partial x_{\nu}} + \frac{\partial \mathfrak{G}^{\mu\nu}}{\partial x_{\mu}} \frac{\partial \varphi^{\alpha}}{\partial x_{\nu}} - \frac{\partial \mathfrak{G}^{\nu\alpha}}{\partial x_{\mu}} \frac{\partial \varphi^{\beta}}{\partial x_{\nu}} \right\} = 0, \quad (17)$$

$$\varphi^{\alpha} = (\partial x'_{\alpha}/\partial \varepsilon)_{\varepsilon=0}$$

are arbitrary functions of the variables x_0, x_1, x_2, x_3 . If arbitrary constants are chosen as φ^α , then from (17) Einstein's conservation laws (10) follow at once. Setting in (17) $\varphi^\alpha = c_\beta \mathfrak{G}^{\alpha\beta}$, where c_β are arbitrary constants, we obtain the conservation laws considered in (6). If, however, using the equations of gravitation, one eliminates the mass tensor from (17), then one arrives at the conservation laws considered in (4,7,8).

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Note: Figure translations are in progress. See original paper for figures.

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