INTEGRAL GEOMETRY ON THE MANIFOLD OF k-DIMENSIONAL PLANES

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.948.5

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR I. M. Gelfand, M. I. Graev, Z. Ya. Shapiro

INTEGRAL GEOMETRY ON THE MANIFOLD OF k-DIMENSIONAL PLANES

Consider the set $H_{n,k}$ of k-dimensional planes of the n-dimensional complex space C^n. On each plane $h \in H_{n,k}$ let us specify a measure μ_h, invariant with respect to parallel translations. Further, let $f(x)$ be an infinitely differentiable and rapidly decreasing function in C^n. The formula

$$\varphi(h) = \int_h f(x)\, d\mu_h \tag{1}$$

assigns to each such function $f(x)$ a certain function of the plane, $\varphi(h)$. The aim of the present work is to invert formula (1). Since the manifold $H_{n,k}$ has dimension greater than n (namely $(k+1)(n-k)$), in order to determine $f(x)$ it is natural to use the values of the function $\varphi(h)$ (and its derivatives) only on some n-dimensional submanifold of $H_{n,k}$. It will also be clarified which class of functions on $H_{n,k}$ is determined by formula (1).

1. We shall assume that the measure μ_h satisfies the following two conditions: 1) if $A \subset h$ is a set of positive measure and g is a parallel translation in C^n, then $\mu_{hg}(Ag) = \mu_h(A)$; 2) if g is an arbitrary affine transformation in C^n, then $\mu_{hg}(Ag)$ is a continuous infinitely differentiable function of g*.

Denote by $G_{n,k}$ the complex Grassmann manifold of k-dimensional subspaces of the space C^n (i.e., k-dimensional planes passing through the point O). Consider the mapping

$$\pi : H_{n,k} \to G_{n,k},$$

which assigns to each plane $h \in H_{n,k}$ the plane from $G_{n,k}$ parallel to it. This mapping endows $H_{n,k}$ with the structure of a fiber space whose base is $G_{n,k}$, and whose fiber is the set of mutually parallel k-dimensional planes. For each point $\beta \in C^n$ we further introduce the mapping

$$s_\beta : G_{n,k} \to H_{n,k},$$

which assigns to each plane from $G_{n,k}$ the parallel plane passing through the point β. Obviously, the composition $\pi \circ s_\beta$ is the identity mapping of the base onto itself.

Denote by Φ_H the set of all infinitely differentiable functions on $H_{n,k}$, and by $\Phi_H^{(p,q)}$ the space of forms of type (p,q) on $H_{n,k}$ with coefficients from Φ_H. Analogously define Φ_G and $\Phi_G^{(p,q)}$. The mappings π and s_β induce mappings of functions (respectively, of differential forms)

$$\pi^* : \Phi_G \to \Phi_H, \quad \Phi_G^{(p,q)} \to \Phi_H^{(p,q)};$$

$$s_\beta^* : \Phi_H \to \Phi_G, \quad \Phi_H^{(p,q)} \to \Phi_G^{(p,q)}.$$

* Such a compatible set of measures can be introduced, for example, by specifying a Hermitian metric in C^n and defining $\mu_h(A)$ as the measure in this metric.

2. In this section we shall construct the operator $\varkappa_\rho$, which plays the principal role in the inversion formula and maps Φ_H into $\Phi_G^{(k,k)}$. Before defining this operator, let us consider the following algebraic construction.

Introduce the space $S^k(C^n) \otimes \bar{S}^k(C^n)$, where $S^k(C^n)$ is the set of symmetric polylinear forms of the first kind of degree k in the space C^n, and $\bar{S}^k(C^n)$ is the set of analogous forms of the second kind. An arbitrary element of this space may be represented in the form

$$B(\xi_1, \dots, \xi_k; \xi_1', \dots, \xi_k'), \qquad \xi_\nu, \xi_\nu' \in C^n,$$

where B is a form of degree $2k$ with complex coefficients, symmetric both in the first and in the second group of arguments, linear in the arguments ξ_ν and antilinear in the arguments ξ_ν'. Now consider two square matrices of order k, whose elements are the vectors $\xi_{ij} \in C^n$ and $\xi_{ij}' \in C^n$ $(i,j = 1, \dots, k)$. To each form $B(\xi_1, \dots, \xi_k; \xi_1', \dots, \xi_k')$ and to the matrices $\Xi = \|\xi_{ij}\|$, $\Xi' = \|\xi_{ij}'\|$ we associate the new form

$$\mathrm{Det}_B(\Xi, \Xi') = \sum_{\sigma,\sigma'} (-1)^{\mu+\nu} B(\xi_{1,s_1}, \dots, \xi_{k,s_k}; \xi_{1,t_1}', \dots, \xi_{k,t_k}'),$$

where the summation is over all permutations $\sigma(s_1, \dots, s_k)$, $\sigma'(t_1, \dots, t_k)$ of the numbers $(1, 2, \dots, k)$, and μ and ν are the numbers of inversions in the permutations σ and σ'. It is easy to see that, like the ordinary numerical determinant, this "determinant of vectors" is skew-symmetric with respect to the rows and with respect to the columns of the matrices Ξ and Ξ'.

Lemma 1. *If the form $B(\xi_1, \dots, \xi_k; \xi'_1, \dots, \xi'_k)$ is defined in the quotient space C^n by some subspace E* and the k vectors $\xi_{i1}, \dots, \xi_{ik}$ or $\xi'_{i1}, \dots, \xi'_{ik}$, for some fixed i, belong to E, then* $\mathrm{Det}_B(\Xi, \Xi') = 0$.

We now proceed to the construction of the operator $\varkappa_\rho$. Let $\varphi(h) \in \Phi_H$. Take an arbitrary fixed point $\beta \in C^n$. The transformation s^*_β takes the function $\varphi(h)$ into a function on $G_{n,k}$, depending on the point β as on a parameter. Put $s^*_\beta \varphi(h) = \varphi(\alpha; \beta)$, where $\alpha = \pi h$, i.e. α is the plane in $G_{n,k}$ parallel to h. Clearly, if β and β' are different points of the plane h, then $\varphi(\alpha; \beta) = \varphi(\alpha; \beta')$. Define now, by means of the function φ, the form

$$B_\varphi(\xi_1, \dots, \xi_k; \xi'_1, \dots, \xi'_k) = d^{k,k}_\beta \varphi(\alpha; \beta) =$$

$$= \sum_{\substack{p_1,\dots,p_k \\ q_1,\dots,q_k}} \frac{\partial^{2k} \varphi(\alpha; \beta)}{\partial \beta_{p_1} \cdots \partial \beta_{p_k} \partial \bar{\beta}_{q_1} \cdots \partial \bar{\beta}_{q_k}} \xi_1^{p_1} \cdots \xi_k^{p_k} \bar{\xi}_1^{q_1} \cdots \bar{\xi}_k^{q_k}.$$

Here the upper index is the number of a coordinate of the vector. The summation is over the values of each of the upper indices from 1 to n. Next assign to an arbitrary point $\alpha \in G_{n,k}$ a frame consisting of k linearly independent vectors $\alpha_j \in C^n$ $(j = 1, \dots, k)$. Consider k arbitrary displacements of this frame and put $\xi_{ij} = \xi'_{ij} = d_i \alpha_j$ $(i, j = 1, \dots, k)$. Define the operator $\varkappa_\rho$ by the formula

$$\varkappa_\rho \varphi(h) = \left(\frac{i}{2}\right)^k \mathrm{Det}_{d^{k,k}_\beta \varphi}(\Xi, \Xi') =$$

$$= \left(\frac{i}{2}\right)^k \sum_{\sigma, \sigma'} (-1)^{\mu+\nu} \sum_{\substack{p_1,\dots,p_k \\ q_1,\dots,q_k}} \frac{\partial^{2k} \varphi(\alpha; \beta)}{\partial \beta_{p_1} \cdots \partial \beta_{p_k} \partial \bar{\beta}_{q_1} \cdots \partial \bar{\beta}_{q_k}} \times$$

$$\times d_1 \alpha^{p_1}_{s_1} \wedge \cdots \wedge d_k \alpha^{p_k}_{s_k} \wedge d_1 \bar{\alpha}^{q_1}_{t_1} \wedge \cdots \wedge d_k \bar{\alpha}^{q_k}_{t_k}.$$

* The form B is defined in the quotient space C^n/E if it vanishes whenever at least one of the vectors ξ_ν belongs to E.

Here σ, σ', as above, are permutations of the numbers $(1, 2, \dots, k)$, and μ and ν are the numbers of inversions in these permutations. Formally, the form $\chi_{\beta\varphi}$ depends on k^2 vectors from C^n. However, when the increment of the vector β lies in the subspace $\alpha \subset C^n$, then $d^{k,k}_\beta \varphi(\alpha; \beta) = 0$, i.e., the form $d^{k,k}_\beta \varphi(\alpha; \beta)$ is defined in the quotient space C^n/α. Hence, and from Lemma 1, it follows easily that the form $\chi_{\beta\varphi}$ is defined in the tangent space to $G_{n,k}$ at the point a, i.e., the following holds.

Lemma 2. The form $\chi_{\beta\varphi}(h)$ belongs to $\Phi^{(k,k)}_G$.

3. We proceed to the formulation of the main theorems.

Theorem 1. In order that a function $\varphi(h)$ on $H_{n,k}$, where $k < n-1$, be representable in the form (1), where $f(x)$ is a rapidly decreasing infinitely differentiable function on C^n, it is necessary and sufficient that the following conditions hold:

1) $\varphi(h) \in \Phi_H$; for every fixed α, the function $s^*_\beta \varphi(h) = \varphi(\alpha; \beta)$ is a rapidly decreasing function in the space C^n/α *.

2) The form $\chi_{\beta\varphi}(h)$ is closed in $\Phi_G^{(k,k)}$ **.

Theorem 2. If $\varphi(h) = \int_h f(x)\, d\mu_h$, and γ is an arbitrary cycle in $G_{n,k}$ of real dimension $2k$, then

$$\int_\gamma \chi_{\beta\varphi} = c_\gamma f(\beta), \tag{2}$$

where the constant c_γ depends only on the homology class of the cycle γ.

Let $\gamma_0, \gamma_1, \dots, \gamma_\nu$ be a basis of the $2k$-dimensional homology group of the complex Grassmann manifold $G_{n,k}$. It is known ([1]) that each of the cycles γ_s is determined by a partition of the number k into a sum of k nonincreasing integers $a_1 + a_2 + \dots + a_k = k,\ 0 \le a_1 \le \dots \le a_k \le n-k$. In particular, the cycle γ_0, corresponding to the partition $1 + 1 + \dots + 1 = k$, consists of all k-dimensional subspaces $G_{n,k}$ belonging to a fixed $(k+1)$-dimensional subspace, i.e. $\gamma_0 \approx G_{k+1,k}$.

Theorem 3. $c_{\gamma_0} = (-1)^k \pi^{2k}/(k!)^2; \quad c_{\gamma_s} = 0,\ s = 1, \dots, \nu.$

The inversion formula (2) can be rewritten in a somewhat different form. Let us specify a k-dimensional plane h by equations of the form

$$P_i(x) \equiv x_i - \alpha_i^1 x_{l+1} - \dots - \alpha_i^k x_n - \beta_i = 0, \qquad i = 1, 2, \dots, l,$$

where $l = n - k$. Define the functional $\varphi(h)$ by the formula

$$\varphi(h) = (\delta(P_1, \dots, P_l), f(x)) \ ***.$$

Then the inversion formula (2) is equivalent to the relation

$$\left(\frac{i}{2}\right)^k \sum_{\sigma,\sigma'} (-1)^{\mu+\nu} \int_\gamma \sum_{\substack{p_1,\dots,p_k \\ q_1,\dots,q_k}} \frac{\partial^{2k} \delta(P_1 - P_1^0, \dots, P_l - P_l^0)}{\partial P_{p_1} \cdots \partial P_{p_k}\, \partial \overline{P}_{q_1} \cdots \partial \overline{P}_{q_k}} \times$$

$$\times d_1 \alpha_{s_1}^{p_1} \wedge \dots \wedge d_k \alpha_{s_k}^{p_k} \wedge d_1 \overline{\alpha}_{t_1}^{q_1} \wedge \dots \wedge d_k \overline{\alpha}_{t_k}^{q_k} = c_\gamma \delta(x - x_0),$$

where $x \in C^n$, x_0 is a fixed point in C^n, $P_i^0 = P_i(x_0)$.

Received
25 III 1966

REFERENCES

[1] Chuan Shen-shen, *Complex Manifolds*, IL, 1961. [2] I. M. Gel' fand, M. I. Graev, N. Ya. Vilenkin, *Integral Geometry and Related Questions of Representation Theory*, Moscow, 1962.

* As was already noted, it follows from the definition of $\varphi(\alpha;\beta)$ that, as a function of β, it is constant on the cosets C^n/α.

** For $k = n-1$ the condition formulated above is insufficient. For a formulation of the necessary and sufficient conditions for $k = n - 1$, see ([2]).

*** $\delta(P_1, \dots, P_l)$ may be defined as the product of generalized functions $\delta(P_1), \dots, \delta(P_l)$; for the definition of the function $\delta(P)$, see ([2]).

Note: Figure translations are in progress. See original paper for figures.