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HYDROMECHANICS

1966

SovietRxiv

Abstract

Full Text

UDC 624.131.5

HYDROMECHANICS

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SOME EXACT SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL (WITH PLANE WAVES) UNSTEADY MOTION OF SOIL

(Presented by Academician L. I. Sedov on 17 XI 1965)

1. Formulation of the problem. One-dimensional unsteady motions of soil with plane waves are described by equations (1)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0;$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0; \quad (1)$$

$$\rho = R(P, D^*);$$

$$\frac{\partial P^*}{\partial t} + u\frac{\partial P^*}{\partial x} = \left[\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}\right] e(P^* - P)e\left(\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}\right).$$

Consider boundary conditions of the form:

$$\text{for } x = h(t) \quad u = \dot{h}(t) = dh/dt, \quad P = P_0\varphi(t) \quad (\varphi(0) = 1);$$

$$\text{for } x = s(t) \quad u = (1 - \rho_0/\rho)\dot{s}(t), \quad P = \rho_0(1 - \rho_0/\rho)\dot{s}^2(t), \quad (2)$$

where $P_0\varphi(t)$ is a prescribed function of time; $h(t)$ and $s(t)$ are a priori unknown functions, whose determination is a constituent part of the problem. We prescribe the initial conditions in the form

$$h(0) = s(0) = 0. \quad (3)$$

It is convenient to formulate the problem in the independent variables τ, l, where l is determined by the equation $dl = \rho/\rho_0(dx - u\,dt)$, $\tau \equiv t$. Introducing the notation $\theta = 1 - \rho_0/\rho$; $F(P, P^*) = 1 - \rho_0/R(P, P^*)$, we rewrite equations (1) in the new variables

$$\frac{\partial\theta}{\partial\tau} + \frac{\partial u}{\partial l} = 0;$$

$$\frac{\partial u}{\partial\tau} + \frac{1}{\rho_0}\frac{\partial P}{\partial l} = 0; \tag{4}$$

$$\theta = F(P, P^*);$$

$$\frac{\partial P^*}{\partial\tau} = \frac{\partial P}{\partial\tau}\, e(P^* - P)\, e\left(\frac{\partial P}{\partial\tau}\right).$$

The boundary and initial conditions (2) and (3) pass into the relations:

$$\text{for } l = l_0 \quad P = P_0\varphi(\tau) \quad (\varphi(0) = 1);$$

$$\text{for } l = S(\tau) \quad u = \theta\dot{S}(\tau), \quad P = \rho_0\theta\dot{S}^2(\tau), \tag{5}$$

where by $S(\tau)$ is denoted the function $s(\tau) + l_0$; l_0 is an arbitrary positive constant.

It is expedient to give equations (4) and (5) a dimensionless form by setting

$$\theta = \theta_0\bar{\theta}, \quad u = (P_0\theta_0/\rho_0)^{1/2}\bar{u}, \quad P = P_0\bar{P}, \quad P^* = P_0\bar{P}^*, \quad l = l_0\bar{l}, \quad \tau = l_0(\rho_0\theta_0/P_0)^{1/2}\bar{\tau},$$

$$F(P, P^*) = \theta_0\bar{F}(\bar{P}, \bar{P}^*); \quad S(\tau) = l_0\bar{S}(\bar{\tau}); \quad \varphi(\tau) = \bar{\varphi}(\bar{\tau}),$$

where $\theta_0 = F(P_0, P_0)$.

The dimensionless form of the equations of the formulated problem will have the form (the overbar over dimensionless quantities is omitted for convenience):

$$\begin{gathered}\frac{\partial\theta}{\partial\tau} + \frac{\partial u}{\partial l} = 0;\\ \frac{\partial u}{\partial\tau} + \frac{\partial P}{\partial l} = 0;\\ \theta = F(P, P^*);\\ \frac{\partial P^*}{\partial\tau} = \frac{\partial P}{\partial\tau}e(P^* - P)e\left(\frac{\partial P}{\partial\tau}\right).\end{gathered} \tag{6}$$

The dimensionless boundary and initial conditions will be written in the form:

$$\begin{gathered} \text{for } l = 1 \qquad P = \varphi(\tau); \\ \text{for } l = S(\tau) \qquad u = \dot{\theta} S = u_s(\tau), \quad P = \dot{\theta} S^2 = P_s(\tau); \\ S(0) = \dot{S}(0) = 1. \end{gathered} \tag{7}$$

2. Seeking self-similar solutions. We shall seek solutions possessing the property that the particle experiences the greatest compressive stress in its entire history once, at $l = S(\tau)$, i.e.,

$$P^* = P^*(l), \qquad P^*(S(\tau)) = P_s(\tau). \tag{8}$$

Let us consider a soil with the following properties:

1) the function $F(P, P^*)$ has the form $\theta = F^*(P^*) f(P/P^*)$, where $F^*(P) = F(P, P)$.
2) $F^*(P) = P^{\cos^2 \mu}$, $\mu = \text{const}$.

Under these assumptions there exist self-similar solutions; to find them we set: $P = P^*(l)\pi$, $u = u^*(l)\vartheta$, $P^*(l) = l^{-2m^2/\sin^2 \mu}$, $u^*(l) = l^{-m^2(2/\sin^2 \mu - 1)}$, $m = \text{const}$, $l = S(\tau)\lambda$, $\pi = \pi(\lambda)$, $\vartheta = \vartheta(\lambda)$.

Then, as is easy to see,

$$\begin{aligned} S(\tau) &= \left[(1+m^2)\tau + 1\right]^{1/(1+m^2)}, \\ P_s(\tau) &= \left[(1+m^2)\tau + 1\right]^{[-2m^2/(1+m^2)](1/\sin^2 \mu)}, \\ u_s(\tau) &= \left[(1+m^2)\tau + 1\right]^{-[m^2/(1+m^2)](2/\sin^2 \mu - 1)}, \end{aligned} \tag{9}$$

and $\pi(\lambda)$ and $\vartheta(\lambda)$ are determined from the ordinary differential equations

$$\begin{aligned} \frac{d\pi}{d\lambda} &= \frac{2m^2}{\lambda \sin^2 \mu} \, \frac{\pi + (1 - 0.5 \sin^2 \mu)\lambda^{1+m^2}\vartheta}{1 - \frac{df}{d\pi}\lambda^{2(1+m^2)}}; \\ \frac{d\vartheta}{d\lambda} &= \left(\frac{2}{\sin^2 \mu} - 1\right) \frac{m^2}{\lambda} \, \frac{\vartheta + \frac{\lambda^{1+m^2} \, df/d\pi}{1 - 0.5 \sin^2 \mu}\pi}{1 - \frac{df}{d\pi}\lambda^{2(1+m^2)}}. \end{aligned} \tag{10}$$

and the initial conditions

$$\pi(1) = \vartheta(1). \tag{11}$$

The function of the surface pressure $\varphi(\tau)$, generating the solution determined above, is represented in the form

$$\varphi(\tau) = \pi\left(\frac{1}{S(\tau)}\right) = \pi\left(\left[(1+m^2)\tau + 1\right]^{-1/(1+m^2)}\right). \tag{12}$$

It is obvious that $\varphi(\tau) < 1$ for $\tau > 0$, if $\pi < 1$ for $0 < \lambda < 1$. Investigation of equations (10) in the general case is a rather difficult problem; however, for a given function $f(\pi)$, the construction of the solution by numerical methods can be done quite simply. In this case

it should be borne in mind that only those solutions of equations (10) have physical meaning which satisfy the condition

$$\pi \leq 1 \quad \text{for} \quad 0 < \lambda \leq 1. \tag{13}$$

At least in two cases, equations (10) are immediately solvable in elementary functions.

1st case: $m = 0$—is trivial and needs no discussion.

2nd case: $df/d\pi = 0$. Equations (10) take the form

$$\begin{aligned} \frac{d\pi}{d\lambda} &= \frac{2m^2}{\sin^2\mu}\frac{\pi}{\lambda} + \left(\frac{2}{\sin^2\mu} - 1\right) m^2 \lambda^{m^2} \vartheta, \\ \frac{d\vartheta}{d\lambda} &= m^2\left(\frac{2}{\sin^2\mu} - 1\right)\frac{\vartheta}{\lambda}. \end{aligned} \tag{14}$$

The solution of these equations under the initial conditions (11) is found in the form

$$\vartheta = \lambda^{m^2(2/\sin^2\mu - 1)};$$

$$\pi = \lambda^{2m^2/\sin^2\mu}\left[1 - m^2\left(2/\sin^2\mu - 1\right)(1-\lambda)\right].$$

It is not difficult to verify that condition (13) is satisfied. This solution corresponds to the case in which the medium is incompressible during unloading.

The author expresses his gratitude to S. S. Grigoryan, under whose supervision the present work was carried out.

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Received
14 XI 1965

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Note: Figure translations are in progress. See original paper for figures.