ON SHARP ESTIMATES OF EIGENFUNCTIONS OF THE BIHARMONIC OPERATOR

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Abstract

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MATHEMATICS

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ON SHARP ESTIMATES OF EIGENFUNCTIONS OF THE BIHARMONIC OPERATOR

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In this note we establish sharp estimates in a closed domain for eigenfunctions of the biharmonic operator and their derivatives. It is known how important such estimates are in questions of expansion in eigenfunctions of self-adjoint operators.

The problem under consideration is as follows: let g be an arbitrary bounded m-dimensional domain with boundary Γ; let $\{u_n(x)\}$ be orthonormal eigenfunctions; and let $\{\lambda_n\}$ be the corresponding eigenvalues of the problem

$$\Delta^2 u - \lambda u = 0 \quad \text{in the domain } g; \qquad u\big|_\Gamma = \partial u/\partial\nu\big|_\Gamma = 0. \tag{1}$$

Here $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator; ν is the outward normal to the boundary Γ; $\lambda_n > 0$; $\lambda_n \to \infty$ as $n \to \infty$. It is required to find an estimate of

$$\max_{x\in(g+\Gamma)} |u_n(x)|.$$

If one takes the m-dimensional ball as the domain g, then direct computation shows that the eigenfunctions of problem (1) possessing radial symmetry take, at the center of the ball—the point x_0—the values $|u_n(x_0)| = C_n\lambda_n^{(m-1)/8}$, where $C_n \geq C > 0$. Therefore, in the case of an arbitrary domain g, an estimate of the maximum modulus of the eigenfunctions of the form*

$$\max_{x\in(g+\Gamma)} |u_n(x)| = O[f(\lambda_n)] \tag{2}$$

is possible only if $f(\lambda_n) \geq \lambda_n^{(m-1)/8}$.

Here I prove that for eigenfunctions of problem (1) in an arbitrary closed domain $g + \Gamma$ the sharp estimate (2) is valid with $f(\lambda_n) = \lambda_n^{(m-1)/8}$, i.e. the estimate

$$\max_{x\in(g+\Gamma)} |u_n(x)| = O\left(\lambda_n^{(m-1)/8}\right). \tag{3}$$

Obtaining this estimate in any interior closed subdomain g' of the domain g presents no serious difficulties, since here one can use considerations analogous to those applied by Titchmarsh in deriving sharp interior estimates for eigenfunctions of the Laplace operator (see (^{1})). Estimates in a closed domain require an entirely different approach. The method I use is close to that applied in works (^{2}, ^{3}) in the study of eigenfunctions of the Laplace operator.

We shall assume that the domain $(g+\Gamma)$ belongs to the class** $A^{(4)}$.

Lemma 1. *For the eigenfunctions of problem (1) the following integral estimates over the domain g hold:*

$$\int_g \sum_{i=1}^{m} \left(\frac{\partial u_n}{\partial x_i}\right)^2 dx = O\left(\lambda_n^{1/2}\right), \qquad \int_g (\Delta u_n)^2\, dx = O(\lambda_n), \qquad \int_g \sum_{i,k=1}^{m} \left(\frac{\partial^2 u_n}{\partial x_i\, \partial x_k}\right)^2 dx = O(\lambda_n). \tag{4}$$

Lemma 1 is a trivial consequence of conditions (1) and of Green' s first and second formulas.

Lemma 2. *For the eigenfunctions of problem 1 the following integral estimates over the boundary Γ are valid:*

$$\int_\Gamma (\Delta u_n)^2\, ds = O(\lambda_n), \qquad \int_\Gamma \left[\frac{\partial}{\partial \nu}(\Delta u_n)\right]^2 ds = O(\lambda_n^{3/2}). \tag{5}$$

The proof of this lemma is cumbersome and cannot be given here in full; we indicate only the main points. The first estimate of the lemma is obtained simply: for any eigenfunction the equality

$$\int_\Gamma (\Delta u_n)^2\, ds = 2\int_g \Delta u_n \sum_{i=1}^{m} \frac{\partial(\Delta u_n)}{\partial x_i}\, q_i\, dx + \int_g (\Delta u_n)^2 \sum_{i=1}^{m} \frac{\partial q_i}{\partial x_i}\, dx =$$

$$= 2\int_g u_n \Delta\left(\sum_{i=1}^{m} \frac{\partial(\Delta u_n)}{\partial x_i}\, q_i\right) dx + \int_g (\Delta u_n)^2 \sum_{i=1}^{m} \frac{\partial q_i}{\partial x_i}\, dx, \tag{6}$$

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* The constants entering the O-term estimates here and below do not depend on n.

** This means that the function defining the equation of the surface Γ in local coordinates belongs to the clas is valid, where $q_i(x)$ are smooth functions satisfying the condition $q_i(x)|_{x\in\Gamma} = \cos(\nu, x_i)$. The right-hand side of formula (6) is easily estimated with the aid of integration by parts, Lemma 1, and conditions (1). As a result we obtain the required estimate

$$\int_\Gamma [(\Delta u_n)^2]\, ds = O(\lambda_n).$$

The proof of the second assertion of Lemma 2 is much more complicated. Its idea is as follows. First, using Green' s first formula, we obtain the equality

$$\int_\Gamma \left[\frac{\partial}{\partial\nu}(\Delta u_n)\right]^2 ds = \int_\Gamma [\nabla_\tau(\Delta u_n)]^2\, ds + O\left(\int_g [\nabla(\Delta u_n)]^2\, dx\right) + O(\lambda_n^{3/2}), \quad (7)$$

where ∇_τ is the gradient in the hyperplane tangent to Γ; ∇ is the gradient. Then, locally straightening the boundary Γ and introducing a cutoff function, we estimate

$$\int_\Gamma [\nabla_\tau(\Delta u_n)]^2\, ds \quad \text{by} \quad O(\lambda_n^{1/2}) \left(\int_\Gamma \left[\frac{\partial}{\partial\nu}(\Delta u_n)\right]^2 ds\right)^{1/2} + O(\lambda_n^{3/2}) +$$

$$+O\left(\int_g [\nabla(\Delta u_n)]^2 dx\right).$$

Taking into account that

$$\int_g [\nabla(\Delta u_n)]^2 dx = -\lambda_n \int_g u_n \Delta u_n\, dx + \int_\Gamma \Delta u_n \frac{\partial(\Delta u_n)}{\partial\nu}\, ds$$

and using the first estimate (5), we find

$$\int_g [\nabla(\Delta u_n)]^2 dx = O(\lambda_n^{3/2}), \qquad \int_g \sum_{i,k,l=1}^m \left(\frac{\partial^3 u_n}{\partial x_i \partial x_k \partial x_l}\right)^2 dx = O(\lambda_n^{3/2}), \quad (8)$$

and then

$$\int_g \left[\frac{\partial}{\partial\nu}(\Delta u_n)\right]^2 ds = O(\lambda_n^{3/2}), \qquad \int_\Gamma [\nabla_\tau(\Delta u_n)]^2 ds = O(\lambda_n^{3/2}). \quad (9)$$

We note that all estimates (4), (5), (8), and (9) are sharp.

Let δ, $\delta \in (0, \delta_0)$, be sufficiently small. Denote by Γ_δ the equidistant surface at distance δ from Γ and lying inside the domain g, and by g^δ the domain bounded by Γ_δ.

Lemma 3. *There exist positive numbers δ, n_0, C_1, C_2 such that, if some eigenfunction $u_n(x)$ of problem (1) with number $n \geq n_0$ attains its maximum value at a point $x_0 \in (g - g^{\delta/2})$, then in the ball $K(x_0, \chi) \subset (g - g^\delta)$ with center at x_0 and radius $\chi = C_1 \lambda_n^{-1/4}$ there are two points x_1 and x_2, lying on one normal to Γ, such that*

$$|u_n(x_1) - u_n(x_2)| \geq \frac{1}{C_2} |u_n(x_0)|.$$

This lemma is not difficult to derive from the mean-value theorem for the equation
$\Delta^2 u - \lambda u = 0$.

Theorem 1. *For the maximum of the modulus of the eigenfunctions of problem (1), the estimate (3) holds uniformly in the closed domain $(g + \Gamma)$.*

Proof. By the Green–Stokes formula, for any eigenfunction $u_n(x)$ we have the equality

$$u_n(x) = \int_\Gamma \left[G(x,y) \frac{\partial(\Delta u_n)}{\partial \nu} - \Delta u_n \frac{\partial G(x,y)}{\partial \nu} \right] ds_y; \tag{10}$$

here $G(x,y)$ is the fundamental solution of the equation $\Delta^2 v - \lambda v = 0$, equal to*

$$C r_{xy}^{(2-m)/2} \lambda_n^{(m-6)/8} \left[\frac{\pi}{2} N_{(m-2)/2} \left(r_{xy} \lambda_n^{1/4} \right) + K_{(m-2)/2} \left(r_{xy} \lambda_n^{1/4} \right) \right].$$

1°. Let $x \in \overline{g}_{\delta/2}$. Since $r_{x\Gamma} \geq \delta/2$, and since for the functions $N_\mu(z)$ and $K_\mu(z)$, for $z \geq 1$, the estimates $|N_\mu(z)| = O(1/\sqrt{z})$ and $|K_\mu(z)| = O((1/\sqrt{z})e^{-z})$ hold, from formula (10) we obtain

$$|u_n(x)| = O\left(\lambda_n^{(m-7)/8} \int_\Gamma \left| \frac{\partial(\Delta u_n)}{\partial \nu} \right| ds + \lambda_n^{(m-5)/8} \int_\Gamma |\Delta u_n| \, ds \right).$$

Hence, and from Lemma 2, the estimate follows

$$\max_{x \in \overline{g}_{\delta/2}} |u_n(x)| = O\left(\lambda_n^{(m-1)/8} \right). \tag{11}$$

2°. Let now $x \in \overline{(g - g_{\delta/2})}$. Differentiate (10) in the direction of the normal to Γ. Taking into account the formula $d(z^{-\mu}N_{\mu}(z))/dz = -z^{-\mu}N_{\mu+1}(z)$ and the analogous formula for $K_{\mu}(z)$, in the right-hand side of (10), after differentiation, we obtain four very similar terms. Let us consider one of them, for example,

$$I(x) = C\lambda_n^{(m-4)/8} \int_{\Gamma} r_{xy}^{(2-m)/2} N_{m/2}\left(r_{xy}\lambda_n^{1/4}\right)\cos(r_{xy}, \nu_x)\frac{\partial(\Delta u_n)}{\partial\nu}\, ds_y. \tag{12}$$

Denote by $z \in \Gamma$ the base of that normal to Γ on which the point x lies, and by ρ the distance from x to the point z: $\rho = r_{xz} = r_{x\Gamma}$. Split Γ into three parts: Γ_1, the part of Γ lying outside the ball $K(z, R)$ with center at the point z and radius R, where $R > 0$ is a fixed number such that $\Gamma \cap K(z, R)$ admits, with respect to a local coordinate system $\xi_1, \dots, \xi_m$ with origin at z, a representation of the form $\xi_m = \zeta(\xi_1, \dots, \xi_{m-1})$; Γ_3, the part of Γ lying inside the ball $K(z, \rho)$; $\Gamma_2 = \Gamma - (\Gamma_1 + \Gamma_3)$. Estimate the integral in (12) separately over each part Γ_i, assuming that $R_0 > R > \delta/2$, where R_0 is the radius of the Lyapunov sphere; we shall also assume that $\rho = r_{x\Gamma} \geq C\lambda_n^{-1/4}$. Then for $N_{m/2}(r_{xy}\lambda_n^{1/4})$ the estimate

$$\left|N_{m/2}\left(r_{xy}\lambda_n^{1/4}\right)\right| = O\left(\frac{1}{r_{xy}^{1/2}\lambda_n^{1/8}}\right)$$

is valid,

$$\left|\int_{\Gamma_1} N_{m/2}\left(r\lambda_n^{1/4}\right) r^{(2-m)/2}\cos(r, \nu_x)\frac{\partial(\Delta u_n)}{\partial\nu}\, ds\right| \leq C\int_{\Gamma_1}\frac{1}{r^{1/2}\lambda_n^{1/8}} r^{(2-m)/2}\left|\frac{\partial(\Delta u_n)}{\partial\nu}\right| ds \leq$$

$$\leq C\lambda_n^{-1/8}\frac{1}{R^{(m-1)/2}}\left\{\int_{\Gamma_1}\left[\frac{\partial(\Delta u_n)}{\partial\nu}\right]^2 ds \cdot \int_{\Gamma_1} ds\right\}^{1/2} \leq C\lambda_n^{5/8}\frac{1}{R^{(m-1)/2}} \leq C\lambda_n^{5/8}; \tag{13}$$

$$\left|\int_{\Gamma_3} N_{m/2}\left(r\lambda_n^{1/4}\right) r^{(2-m)/2}\cos(r, \nu_x)\frac{\partial(\Delta u_n)}{\partial\nu}\, ds\right| \leq C\int_{\Gamma_3}\frac{1}{r^{1/2}\lambda_n^{1/8}} r^{(2-m)/2}\left|\frac{\partial(\Delta u_n)}{\partial\nu}\right| ds \leq$$

$$\leq C\lambda_n^{-1/8}\frac{1}{\rho^{(m-1)/2}}\left\{\int_{\Gamma_3}\left[\frac{\partial(\Delta u_n)}{\partial\nu}\right]^2 ds \cdot \int_{\Gamma_3} ds\right\}^{1/2} \leq C\lambda_n^{5/8}\frac{1}{\rho^{(m-1)/2}}\rho^{(m-1)/2} = C\lambda_n^{5/8}. \tag{14}$$

In the estimates (13) and (14) we have used (5) and the fact that for $y \in \Gamma_1$, $r_{xy} \geq R$, while for $y \in \Gamma_3$, $r_{xy} \geq \rho/2$.

Let τ denote the polar radius in the hyperplane tangent to Γ at the point z. Since $(g+\Gamma) \in A^{(4)}$, for $y \in \Gamma \cap K(z,R)$, $|\xi_m| = |\xi(\xi_1, \dots, \xi_{m-1})| \leq C\tau^2$. Obviously,

$$|\cos(r,\nu_x)| \leq (\rho + |\xi_m|)/\tau \leq \rho/\tau + C\tau.$$

Taking this into account, we estimate the integral over Γ_2:

$$\left|\int_{\Gamma_2} N_{m/2}(r\lambda_n^{1/4})\, r^{(2-m)/2} \cos(r,\nu_x) \frac{\partial(\Delta u_n)}{\partial \nu}\, ds\right| \leq C\lambda_n^{-1/8} \int_{\Gamma_2} \left|\frac{\partial(\Delta u_n)}{\partial \nu}\right| r^{(1-m)/2} \left(\frac{\rho}{\tau} + C\tau\right) ds$$
$$\leq C\lambda_n^{-1/8} \left\{ \int_{\Gamma_2} \left[\frac{\partial(\Delta u_n)}{\partial \nu}\right]^2 ds \cdot 2 \left[\int_{\Gamma_2'} \tau^{1-m} \frac{\rho^2}{\tau^2}\, ds' + \int_{\Gamma_2'} \tau^{1-m} C^2 \right.\right.$$
$$\leq C\lambda_n^{1/8} \left\{ \rho^2 \int_\rho^R \tau^{-m-1}\tau^{m-2}\, d\tau + \int_\rho^R \tau^{3-m}\tau^{m-2}\, d\tau \right\}^{1/2} \leq C\lambda \tag{15}$$

the primes in the second inequality denote passage to integration over the hyperplane tangent at the point z.

Comparing (12), (13), (14), and (15), we obtain

$$|I(x)| = O\left(\lambda_n^{(m+1)/8}\right).$$

Since the remaining three terms obtained by differentiating formula (10) are estimated similarly, we finally find

$$|\partial u_n(x)/\partial \nu| = O\left(\lambda_n^{(N+1)/8}\right), \qquad x \in (g - g_{\delta/2}), \quad \rho \geq C\lambda_n^{-1/4}. \tag{16}$$

Let x_0 be the point at which $|u(x)|$ attains its maximum, $U_0 = |u_n(x_0)|$; let z be the point at which $|\partial u_n(x)/\partial \nu|$ attains its maximum, $U_1 = |\partial u_n(z)/\partial \nu|$; and let $y_1(y_2) \in \Gamma$ be the foot of the normal to Γ on which $x_0(z)$ lies. a) If

$$r_{x_0\Gamma} \geq 2\chi = 2C_1\lambda_n^{-1/4},$$

then, applying Lagrange' s formula to the two points mentioned in Lemma 3 and taking (16) into account, we obtain estimate (3). In (16), in this case, C must be taken to be C_1, i.e. $\rho \geq C_1\lambda_n^{-1/4}$. b) If

$$r_{x_0\Gamma} < 2\chi = 2C_1\lambda_n^{-1/4},$$

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* All constants entering the formulas and independent of the number of the eigenfunctions are denoted by the

then

$$U_0 = |u_n(x_0) - u_n(y_1)| = |\partial u(\xi)/\partial\nu|\, r_{x_0\Gamma},$$

i.e. $U_0 \le U_1 r_{x_0\Gamma}$. Therefore

$$U_1 = |\partial u_n(z)/\partial\nu - \partial u_n(y_2)/\partial\nu| = |\partial^2 u_n(\eta)/\partial\nu^2|\, r_{z\Gamma} \le C_0\lambda_n^{1/2} U_0 r_{z\Gamma} \le C_0\lambda_n^{1/2} r_{z\Gamma} r_{x_0\Gamma} U_1$$

(the penultimate inequality follows from Schauder' s estimate for problem (1)). Hence

$$r_{z\Gamma} \ge (C_0\lambda_n^{1/2} r_{x_0\Gamma})^{-1} > (2C_1C_0\lambda_n^{1/4})^{-1},$$

i.e. $\max_{x\in(g-g_{\delta/2})} |\partial u(x)/\partial\nu|$ is attained at a point z such that

$$r_{z\Gamma} > (2C_1C_0)^{-1}\lambda_n^{-1/4}.$$

Therefore, in (16), C must be taken to be $(2C_1C_0)^{-1}$. Estimate (3) is obtained by applying Lagrange' s formula to the points x_0 and y_1, taking into account (16) and the fact that

$$r_{x_0y_1} < 2C_1\lambda_n^{-1/4}.$$

Theorem 2. For the derivatives of the eigenfunctions of problem (1), the following estimate holds uniformly in the closed domain $(g+\Gamma)$:

$$\max_{x\in(g+\Gamma)} |D^k u_n(x)| = O\left(\lambda_n^{(m-1)/8+k/4}\right), \qquad k = 0, 1, 2, \dots . \tag{17}$$

Estimate (17) is sharp. Theorem 2 follows easily from Theorem (1), Schauder' s estimates, and interpolation inequalities (see [4]).

Remark. Estimates (3) and (17) are also valid in the case when, in problem (1), the boundary conditions are taken to be

$$\Delta u|_\Gamma = 0, \qquad \partial(\Delta u)/\partial\nu|_\Gamma = 0.$$

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Note: Figure translations are in progress. See original paper for figures.

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