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# ASYMPTOTICS IN THE GENERAL HARDY- LITTLEWOOD PROBLEM

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ASYMPTOTICS IN THE GENERAL HARDY-LITTLEWOOD PROBLEM

1. In this note we shall present a method for finding the asymptotics of the number of solutions of the equation

$$p + \varphi(\xi, \eta) = n, \quad (1)$$

where  $p$  runs through the prime numbers;  $\varphi(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$  is a given positive quadratic form with discriminant different from a perfect square. Equation (1) is a natural generalization of the Hardy-Littlewood equation, obtained from (1) when  $\varphi(\xi, \eta) = \xi^2 + \eta^2$  and studied in papers <sup>(1,2)</sup>.

In paper <sup>(3)</sup> the solvability of equation (1) for sufficiently large  $n$  was proved, and a lower estimate was obtained for the number of solutions of (1). However, the question of the existence of an asymptotic formula for the number of solutions of (1) remained open.

The main difficulty lay in the fact that the general form  $\varphi(\xi, \eta)$ , generally speaking, is multiclass; in this connection, we know how to solve the equation  $\varphi(\xi, \eta) = m$  only for the case when the values  $\varphi(\xi, \eta)$  run through an entire genus of quadratic forms. The passage to an individual form was therefore carried out with the loss of a certain percentage of the asymptotics.

In note <sup>(4)</sup> some ergodic properties of the solutions of the classical Hardy-Littlewood equation were considered. Similar properties are found for the general equation (1), as a result of which it is possible to find the asymptotics for the number of solutions of this equation.

2. Let us derive the asymptotics for the number  $Q(n)$  of solutions of equation (1) in the case when  $\varphi(\xi, \eta)$  is a primitive form with negative discriminant  $-d = b^2 - 4ac = -2^\alpha P$ , where  $\alpha = 0, 2$ , or  $3$ , and  $P$  is the product of distinct odd prime numbers. Under the indicated conditions one has

**Theorem 1.** *If even  $n \rightarrow \infty$ , then*

$$Q(n) = C_\varphi(n) A_d(n) \frac{n}{\ln n} (1 + \varepsilon(n)), \quad (2)$$

where  $C_\varphi(n) \geq c_0 > 0$ , the constant  $c_0$  depends only on the given form  $\varphi(\xi, \eta)$ ,

$$A_d(n) = \frac{2\pi}{\sqrt{d}} \prod_p \left( 1 + \frac{\chi_d(p)}{p(p-1)} \right) \prod_{p|2dn} \frac{(p-1)(p-\chi_d(p))}{p^2 - p + \chi_d(p)},$$

$\chi_d(m)$  is the Kronecker symbol,  $\varepsilon(n) = O((\ln \ln n)^{-\eta})$ ,  $\eta$  is a certain constant subject to the condition  $0 < \eta < 1$ .

In what follows, equation (1) will be considered under the restrictions introduced above, which are not essential.

From the elements of the theory of quadratic forms it follows that equation (1) can be replaced by the equation

$$p + r^2 \sigma N(\mathfrak{a}) = n, \quad (3)$$

where  $p$  is prime,  $r = \prod_{p|P} p^{\alpha_p}$ ,  $\alpha_p \geq 0$ ,  $\sigma$  is a divisor of  $P$ , and, for fixed  $\sigma$ ,  $\mathfrak{a}$  runs through the integral ideals of a certain fixed class  $C_0 = C_0(\sigma)$  of ideals of the quadratic field  $K(\sqrt{-d})$ ,  $N(\mathfrak{a})$  is the norm of the ideal  $\mathfrak{a}$ ,  $(N(\mathfrak{a}), 2d) = 1$ .

In this case

$$Q(n) = w \sum_{\sigma|P} Q_\sigma(n), \quad (4)$$

where  $Q_\sigma(n)$  is the number of solutions of equation (3) for fixed  $\sigma$ ;  $w = 2$  for  $d > 4$ ;  $w = 4$  for  $d = 4$ ;  $w = 6$  for  $d = 3$ .

Let us now consider equation (3) with fixed  $\sigma$  and with  $\mathfrak{a}$  ranging over integral ideals of the genus  $R = R(\sigma)$  containing the class  $C_0$ . Denote the number of solutions of the resulting equation by  $\overline{Q}_\sigma(n)$ .

**Theorem 2.** *If even  $n \rightarrow \infty$ , then*

$$\sum_{\sigma|P} \overline{Q}_\sigma(n) = \frac{t}{w} C_\varphi(n) A_d(n) \frac{n}{\ln n} + O\left(\frac{n}{(\ln n)^{1.042}}\right), \quad (5)$$

where  $C_\varphi(n)$  and  $A_d(n)$  are defined in (2), and  $t$  is the number of ideal classes in the genus.

Theorem 2 is proved by the dispersion method according to the scheme developed in (3, 5).

**Theorem 3.** *If even  $n \rightarrow \infty$ , then*

$$Q_\sigma(n) = \frac{1}{t} \overline{Q}_\sigma(n) (1 + \varepsilon(n)), \quad (6)$$

where  $\varepsilon(n)$  is defined in (2).

Theorem 1 follows immediately from (4), (5), and (6). Thus, the proof of Theorem 1 is essentially reduced to the proof of Theorem 3.

3. Let us consider a brief outline of the proof of Theorem 3. Transform

$$\bar{Q}_\sigma(n) = \sum_{p+r^2\sigma m=n} f(m), \quad (7)$$

where

$$f(m) = \sum_{N(\mathfrak{a})=m, \mathfrak{a} \in R} 1 = \sum_{xy=m} \chi_d(x).$$

The set of numbers  $m$  satisfying (7) will be divided into two classes. In class  $A$  we include those  $m$  whose factorization contains at least  $K_0$  prime numbers  $p_{ij}$  ( $j = 1, 2, \dots, K_0$ ), exactly to the first power, such that the conditions

$$\chi_d(p_{ij}) = 1, \quad p_{ij} = \mathfrak{p}_{ij}\mathfrak{p}'_{ij}, \quad \mathfrak{p}_{ij} \in C_i$$

are fulfilled for each  $i = 1, 2, \dots, h$ , where  $\mathfrak{p}_{ij}$  are prime ideals of the field  $K(\sqrt{-d})$ ,  $h$  is the number of classes of this field,  $K_0 = [(\ln \ln n)^{1-\eta}]$  ( $\mathfrak{p}'_{ij}$  is the conjugate ideal). To class  $B$  we assign the remaining  $m$ .

Thus,

$$\bar{Q}_\sigma(n) = \sum_{m \in A} \Sigma_A + \sum_{m \in B} \Sigma_B. \quad (8)$$

Estimating the sum  $\Sigma_B$  from above by sieve methods (<sup>6-8</sup>), we obtain

$$\Sigma_B = O\left(K_0 \frac{n(\ln \ln n)^{K_0+3}}{(\ln n)(\ln n)^{1/h}}\right). \quad (9)$$

For  $m \in A$  there already holds an asymptotic uniformity of the distribution of integral ideals  $\mathfrak{a}$  satisfying the equation

$$N(\mathfrak{a}) = m, \quad (10)$$

over the ideal classes of the genus  $R$ .

Indeed, in this case

$$\mathfrak{a} = \left( \prod_{\substack{i=1,2,\dots,h \\ j=1,2,\dots,K_0}} \mathfrak{a}_{ij} \right) \mathfrak{a}, \quad (11)$$

where, obviously,  $\gcd(N(\prod_{i,j} \mathfrak{a}_{ij}), N(\mathfrak{a})) = 1$ .

We use the fact that the group  $G_0$  of classes of the principal genus is abelian and therefore can be decomposed into a direct product of cyclic subgroups, namely

$$G_0 = \prod_{s=1}^{s_0} g_s,$$

where  $g_s$  has order  $p_s^{\alpha_s}$ , and each class  $D_s$  generating the group  $g_s$  is represented in the form

$$D_s = C_t^2, \quad t = t(s),$$

where  $C_t$  is some class of the field  $K(\sqrt{-d})$ .

It follows from (11) that from  $\mathfrak{a}$  one can extract prime ideals of the first degree

$$\pi_{i1} \in C_{t(i)}, \quad \pi_{i2} \in C_{t(i)}, \quad \pi_{i3} \in D_i, \quad \pi_{i4} \in D_i^2, \dots, \pi_{iK_1} \in D_i^{2^{K_1-3}},$$

where  $i = 1, 2, \dots, s_0$ ;  $K_1 = K_0/h$ .

As a result we obtain

$$\mathfrak{a} = \left( \prod_{\substack{i=1,2,\dots,s_0 \\ j=1,2,\dots,K_1}} \pi_{ij} \right) \mathfrak{a}_1, \quad (12)$$

where all  $\pi_{ij}$  are distinct and coprime,  $(N(\prod_{i,j} \pi_{ij}), N\mathfrak{a}_1) = 1$ .

Performing in (12) all transformations

$$\pi_{ij} \rightarrow \pi'_{ij}, \quad \pi_{ij} \rightarrow \pi''_{ij}, \quad \pi_{ij_1} \pi_{ij_2} \rightarrow \pi'_{ij_1} \pi'_{ij_2},$$

and so on, we obtain all solutions of equation (10) for fixed  $q_1$ . In this process  $\mathfrak{a}$  will be transformed into solutions belonging to the classes

$$C_j = \left( \prod_{s=1}^{s_0} C_{t(s)}^{\pm 1 \pm 1 + 2 + 2^2 + \dots + 2^{K_1-2}} \right) C(q_1), \quad (13)$$

where  $C_j \in R$ , and  $C(q_1)$  is a fixed class.

The set of multipliers at  $C(q_1)$  will be distributed asymptotically uniformly over all classes of the principal genus. Consequently,  $C_j$  in (13) will also, with asymptotic uniformity, run through all classes of the genus  $R$ . Hence we are able to derive an estimate for the number  $f_{C_0}(m)$  of solutions of equation (10) falling into a prescribed class  $C_0$ .

We obtain

$$f_{C_0}(m) = \frac{f(m)}{t} \left( 1 + O \left( \frac{1}{(\ln \ln n)^\eta} \right) \right), \quad (14)$$

where  $m \in A$ .

From (7), (8), (9), and (14) we derive (6), which completes the proof of Theorem 3.

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