ON THE QUESTION OF HARMONIC ANALYSIS OF BOUNDED SOLUTIONS OF OPERATOR EQUATIONS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.948.35

MATHEMATICS

V. V. ZHIKOV

ON THE QUESTION OF HARMONIC ANALYSIS OF BOUNDED SOLUTIONS OF OPERATOR EQUATIONS

(Presented by Academician I. G. Petrovskii on 22 I 1966)

It is known that in a finite-dimensional space every bounded solution of the equation $u' = B(t)u$ with periodic operator $B(t)$ is an almost-periodic function. The bounded solutions of the inhomogeneous equation $u' = B(t)u + f(t)$ are also almost-periodic if $f(t)$ is an almost-periodic function. This is a consequence of Floquet theory. Although in an infinite-dimensional space Floquet theory remains a problem, nevertheless the structure of bounded solutions can be clarified in a number of cases. This note is devoted mainly to the study of parabolic operator equations. As an example one may consider a parabolic equation with a strongly elliptic operator in a bounded domain.

I. Let H be a Hilbert space and let A be a linear operator with completely continuous resolvent, which is the infinitesimal operator of a strongly continuous semigroup.

Lemma 1. *Every defined and bounded on the whole axis solution u of the equation*

$$u' = Au$$

belongs to the closed linear span of the proper elements of the operator A and is a weakly almost-periodic function.

The proof is carried out according to the following plan. Without loss of generality one may assume that zero is not a point of the spectrum of the operator A. Replacing the bounded solution $u(t)$ by the bounded solution $v(t) = A^{-1}u(t) = Bu(t)$, we shall have

$$Bv' = v. \tag{1}$$

Denote by $D(H, -\infty, \infty)$ the set of all infinitely differentiable finite functions with values in H, and let Φ be the Fourier transform of the function $v(t)$ (in the sense of L. Schwartz’ s theory of distributions). Then, by virtue of (1), for every $\varphi \in D(H, -\infty, \infty)$ we have

$$\langle \Phi, i\lambda B\varphi - \varphi \rangle = 0. \tag{2}$$

Let $x(t) \in D(H, -\infty, \infty)$ and vanish in some neighborhood of the spectrum of the operator A and of zero. Substituting in (2) as the test function

$$\frac{1}{it} R\left(\frac{1}{it} B\right) x(t),$$

we obtain $\langle \Phi, x(t) \rangle = 0$. Hence it follows that Φ is concentrated at the points of the spectrum of the operator A. Further one uses the known theorems on the summation of the Fourier series of a function whose spectrum has the single limit point at infinity (^{1}).

Let $f(t)$ be an almost-periodic function. A consequence of Lemma 1 is the following theorem:

Theorem 1. *Every compact solution of the equation*

$$u' = Au + f$$

is an almost-periodic function.

Indeed, it follows from Lemma 1 that nontrivial bounded solutions of the homogeneous equation cannot be arbitrarily small in norm. Applying the Favard—Amerio theory (^{2}), we obtain that the minimal solution is almost periodic. The assertion of Theorem 1 follows from this.

In particular, Theorem 7 of our note (^{3}) remains valid if one removes from its hypotheses the requirement, contained there, of completeness of the system of eigen- and associated elements.

In the case when $\operatorname{Re}(Axx) \geqslant 0$, Theorem 1 was obtained in (^{4}) with the use of special theorems of the contraction theory of Hilbert space.

Remark. Lemma 1 and Theorem 1 remain valid if the spectrum of the operator A has no more than a finite number of limit points in each finite interval of the imaginary axis.

Our further exposition, concerning parabolic equations periodically dependent on time, is based on the following lemma, which is the discrete analogue of Lemma 1.

Lemma 2. *If on the unit circle there are no more than a finite number of points of the continuous spectrum of a bounded operator T, then from the boundedness of the sequence of the form $\{T^n x_0\}_{n=-\infty}^{\infty}$ it follows that x_0 belongs to the closed linear hull of the eigenvectors of the operator T.*

II. Consider the equation

$$u' + S_1(t)u + S_2(t)u = f(t), \tag{3}$$

where $f(t)$ is an almost-periodic function, and the operators $S_1(t)$ and $S_2(t)$ satisfy the following conditions:

1) $S_1(t+\omega) = S_1(t);\quad S_2(t+\omega) = S_2(t)$;
2) $S_1(t)$ is a self-adjoint operator and $(S_1(t)xx) \geqslant \gamma(xx)$ with a positive constant γ;
3) $\|S_1(t_1)S_1^{-1}(t_2)\| \leqslant C$ for any t_1 and t_2, and the operator $S_2(t)$ is subordinate to the operator $S_1(t)$ in the sense of the solvability theory of equations of parabolic type (^{5}).

Denote by T the monodromy operator, i.e., the operator which assigns to an element $h_0 \in H$ the value at the point ω of the solution of the homogeneous equation (3).

Theorem 2. *If $S_1^{-1}(0)$ is a completely continuous operator, then every bounded solution of equation (3) is an almost-periodic function.*

First it is established that bounded solutions of the homogeneous equation cannot be arbitrarily small. Here Lemma 2 is used (it is not difficult to see that the monodromy operator is completely continuous). Application of Favard' s theory gives the assertion of Theorem 2 with almost periodicity replaced by weak almost periodicity. The proof of strong almost periodicity is carried out according to the same scheme as in Theorem 2 of our note (^{6}).

Additional difficulties arise when the principal operator $S_1(t)$ has continuous spectrum. We describe here two cases: 1) when on the unit circle there lie no more than a finite number of points of the continuous spectrum of the operator T; 2) when the operator is a contraction. In the first case we have, in principle, the same situation as in Theorem 2. Peculiar difficulties arise only because the set of bounded sequences of the form $\{T^n x_0\}$ may be infinite-dimensional. In the second case, almost periodicity holds only for compact solutions.

Theorem 3. *Let $S_1(t) \equiv S(0) = A$; $(Axx) \geqslant 0$ and (for some $c > 0$ and $0 \leqslant \theta < 1$) $(A + cE)^{-\theta}S_2(t) = B(t)K$, where K is a completely continuous operator, some power of which is a Hilbert–Schmidt operator, and $B(t)$ is a strongly continuous operator-valued function. Then every bounded solution of the homogeneous equation (3) and every compact solution of the nonhomogeneous equation (3) are almost-periodic functions.*

Under the hypotheses of Theorem 3 the monodromy operator T is equal to $e^{-\omega A} + L$, where L is a completely continuous operator. Hence, by a well-known theorem of H. Weyl, it follows that the operator T satisfies the hypotheses of Lemma 2. Let us briefly explain why in Theorem 3 boundedness is required of the solution of the homogeneous equation, whereas compactness is required of the solutions of the nonhomogeneous equation. First, one proves the reducibility of equation (3) on the subspace obtained by taking the closure of the set of initial data of bounded solutions. Here the theory of Yu. I. Lyubich and V. I. Macaev (^{7}) is used. From this partial reducibility it follows that, up to a periodic transformation, a bounded solution of the homogeneous equation is a weakly almost-periodic function with a spectrum having a single limiting point at a finite distance. Strong almost-periodicity is then established by means of Dini' s theorem for a monotone sequence of almost-periodic functions (^{8}).

All the hypotheses of Theorem 3 are fulfilled for the equation

$$u_t = \Delta u + p(x,t)u + f(x,t),$$

considered in the whole space under the condition that the function $p(x,t)$ is continuous in both arguments, finite with respect to x, and periodic with respect to t.

Theorem 4. *Let the monodromy operator T be a contraction. Then every compact solution of equation (3) is an almost-periodic function.*

In the proof of Theorem 4 one uses the well-known theorem on the decomposition of a contraction operation and its powers into unitary and completely nonunitary components (^{9}).

Let us consider several other applications of Lemmas 1 and 2. Let $\{A_k\}_{k=0}^{n-1}$ be completely continuous operators. It is proved that every bounded solution of the equation

$$y^n + \sum_{k=0}^{n-1} A_k y^{(k)}$$

is almost-periodic and has almost-periodic derivatives up to order $(n-1)$. This is a generalization of the classical Bohr–Neugebauer results concerning ordinary differential equations with constant coefficients. We note that the operators $\{A_k\}_{k=0}^{n-1}$ may be taken to depend periodically on time (with a common period).

The second application concerns an equation with delayed argument, which in vector notation has the form

$$x' = \sum_{k=1}^{N} A_k(t)x(t+\theta_k) + F(t), \tag{4}$$

where $\{A_k\}_{k=1}^{N}$ are periodic linear operators with a common period in a finite-dimensional space, and $\theta_k \geq 0$. It is asserted that every solution, defined on the whole axis and bounded, is almost-periodic.

In connection with the study of bounded solutions of equation (4), we have obtained a general form of the minimax method. We shall now briefly discuss this question.

III. In a separable Banach space B consider the equation

$$u' = Au + f(t), \tag{5}$$

where $f(t)$ is an almost-periodic function in B. Suppose that the following conditions are satisfied.

1) A is the infinitesimal operator of a strongly continuous subgroup;
2) every nontrivial compact solution of the homogeneous equation cannot be arbitrarily small.

Suppose it is known that there exists a compact solution of equation (5). Does there exist an almost-periodic solution? In the case of a finite-dimensional space, a positive answer is given by the minimax method, due to Favard. L. Amerio' s widely developed generalization of this method applies only to uniformly convex spaces.

Theorem 5. *If there exists a compact solution of equation (5), then there also exists an almost-periodic solution.*

We briefly outline the proof. First of all, instead of the compact solution $u(t)$, we consider a Birkhoff-recurrent solution $v(t)$ (the existence of such a solution can be proved very simply (^{10})). Realize the original space B as a subspace of the space $C(0,1)$, and let Δ denote the closed convex hull of the set of values of $v(t)$. Denote by Ω_f the set of solutions of equation (5) whose trajectories are contained in Δ, and let

$$\delta(x(t)) = \sup_{-\infty<t<\infty} \|x(t)\|_{L_2}$$

for every $x(t) \in \Omega_f$. A solution $\dot{x}(t)$ is called minimal if

$$\delta(x) = \min_{x(t)\in\Omega_f} \delta(x).$$

It is proved that the minimal solution exists, is unique, and is almost-periodic.

Theorem 5 remains valid if the operator A depends on time almost-periodically.

I express my gratitude to Prof. B. M. Levitan for discussing the results of this note.

Moscow State University
named after M. V. Lomonosov

Received
5 I 1966

CITED LITERATURE

[1] B. M. Levitan, *Almost-periodic functions*, Moscow, 1953.
[2] L. Amerio, Ric. Math., **10**, Fasc. 1–2 (1961).
[3] V. V. Zhikov, DAN, **163**, No. 3 (1965).
[4] C. Foias, S. Zaidman, Ann. Scuola Norm. Super. Pisa, Ser. 3, **15**, No. 3 (1961).
[5] O. A. Ladyzhenskaya, Matem. sborn., **39**, issue 4, 491 (1956).
[6] V. V. Zhikov, DAN, **165**, No. 5 (1966).
[7] Yu. L. Lyubich, V. I. Matsaev, Matem. sborn., **56**, No. 4 (1962).
[8] L. Amerio, Rend. Seminario Mat. Univ. Padova, **30**, 288 (1960).
[9] B. Sz. Nagy, S. Fois, Acta Sci. Math., **21**, No. 3–4, 251 (1960).
[10] B. M. Millionshchikov, DAN, **161**, No. 1 (1965).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.