ON THE APPROXIMATION OF A FUNCTION BY RATIONAL FRACTIONS

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Abstract

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ON THE APPROXIMATION OF A FUNCTION BY RATIONAL FRACTIONS

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In this note conditions are formulated on a function under which it can be represented as the limit of a uniformly convergent sequence of rational fractions on a certain prescribed set.

We denote by τ the plane of the complex variable $z = x + iy$; by Ce the complement of the set e in τ; by $A(e, m)$ the set of all functions each of which is analytic outside some closed subset of the set e, is bounded there in modulus by the constant m, and is equal to zero at infinity;
$\gamma(e, f) = \lim_{z\to\infty} zf(z)$, where $f(z) \in A(e, m)$;
$\gamma(e) = \sup_{f\in A(e,1)} |\gamma(e, f)|$ is the analytic capacity of the set e;
$\varphi(z, e)$ is the Ahlfors function for the closed set e, i.e. $\varphi(z, e) \in A(e, 1)$ and $\gamma(e, \varphi) = \gamma(e)$;
$K(z, \delta)$ is the square whose sides are parallel to the coordinate axes, with center at z and side length δ; ∂K is the boundary of the set K.

Theorem. Let e be a closed bounded set; let $f(z)$ be a function continuous in the whole plane and such that, for every square $K(z, \delta)$,

$$\left|\int_{\partial K(z,\delta)} f(\zeta)\, d\zeta\right| \le \gamma[Ce \cap K(z, r\delta)]\, \Omega(\delta),$$

where $r \ge 1$ does not depend on z and δ; $\lim_{\delta\to 0} \Omega(\delta) = 0$. Then $f(z)$ can be uniformly approximated with arbitrary accuracy on the set e by a rational fraction∗.

Lemma 1. Let α be a natural number; $c_{k,n} \le c$ $(k = 1, 2, \dots;\ n = 1, 2, \dots, k\alpha)$ be nonnegative numbers. Then

$$\sum_{k,n} c_{k,n} \ge B(\alpha)c \left[\sum_{k,n} \frac{c_{k,n}}{kc}\right]^2,$$

where $B(\alpha)$ depends only on α.

Lemma 2. Let $\{e_i\}$ be a system of closed subsets of e such that, for every square $K(z, \gamma(e))$, the number of sets of this system intersecting it is not greater than p. Then for all functions $\{f_i(z) \in A(e_i, 1)\}$ the inequality

$$\sup_{z \in Ce} \sum_i |f_i(z)| \leq c(p)$$

holds.

Proof. Fix a point ζ and renumber $\{e_i\}$ with two indices k, n so that $k = [\rho^{-1}(\zeta, e_{k,n})\gamma(e)]$, where $\rho(\zeta, e_{k,n})$ is the distance from the point ζ to $e_{k,n}$. We obtain that the index n runs through no more

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∗ *Remark in proof.* The condition of the theorem is also a necessary condition for approximability of a function by rational fractions. The proof of this fact has been submitted for publication.

$ak = a(p)k$ values. Put

$$\mu(\zeta) = \sum_{k,n} \frac{\gamma(e_{k,n})}{k\gamma(e)}; \qquad \mu = \sup_\zeta \mu(\zeta); \qquad \varphi(z) = \sum_i \varphi(z, e_i),$$

Since for every $g(z) \in A(e, 1)$, $|g(z)| \leq \min\{1, \gamma(e)/\rho(z, e)\}$, it follows that

$$\sup_z |\varphi(z)| \leq C_1(p) \left[1 + \sum_{k,n} \frac{\gamma(e_{k,n})}{k\gamma(e)}\right] \leq C_1(p)(1 + \mu) = \mu_1.$$

Since $\sum_i \gamma(e_i) \leq \mu_1 \gamma(e)$ and since, by Lemma 1, $\sum_i \gamma(e_i) \geq B(\alpha)\gamma(e)\mu^2$, we have $B(\alpha)\mu^2 \leq \mu_1$, i.e. $\mu \leq C_2(p)$. Therefore

$$\sum_i |f_i(\zeta)| \leq C_1(p)(1 + \mu) = C(p).$$

The lemma is proved.

Lemma 3. If $\{e_i\}$ satisfies the conditions of Lemma 2, then

$$\sum_i \gamma(e_i) \leq C(p)\gamma(e).$$

Lemma 4. If $f(z)$ satisfies the conditions of the theorem, then for all z and δ

$$\left|\int_{\partial K(z,\delta)} f(\zeta)(\zeta - z)\,d\zeta\right| \leq \delta\gamma(Ce \cap K(z,\delta))\Phi(\delta),$$

and $\Phi(\delta) \to 0$ as $\delta \to 0$.

Proof. Fix a natural number q and consider the system of squares $K^s_\beta = K(z^s_\beta, \delta q^{-s})$ ($\beta = (\beta_0, \dots, \beta_s)$; $\beta_i = 1, \dots, q^2$; $s = 0, 1, 2, \dots$) such that $K^0_\beta = K(z, \delta)$, for every s any two squares from $\{K^s_\beta\}$ are congruent, and for all s and β

$$K^s_\beta = \bigcup_{\beta_{s+1}=1}^{q^2} K^{s+1}_\beta.$$

We shall denote by the sign $\sum'_\beta$ (respectively $\sum''_\beta$) summation over all those values of β for which $\gamma(Ce \cap K^s_\beta) \geq q^{-1}\delta q^{-s}$ (respectively summation over all remaining values of β). For all s, β,

$$\int_{\partial K^{s-1}_\beta} f(\zeta)(\zeta - z^{s-1}_\beta)\,d\zeta = \sum_{\beta_s}(z^s_\beta - z^{s-1}_\beta)\int_{\partial K^s_\beta} f(\zeta)\,d\zeta +$$

$$+\sum_{\beta_s}{}' \int_{\partial K^s_\beta} f(\zeta)(\zeta - z^s_\beta)\,d\zeta + \sum_{\beta_s}{}'' \int_{\partial K^s_\beta} f(\zeta)(\zeta - z^s_\beta)\,d\zeta;$$

therefore, for every n,

$$\int_{\partial K^0_\beta} f(\zeta)(\zeta - z)\,d\zeta = \sum_{s=1}^{n-1}\sum_{\beta_0,\dots,\beta_{s-1}}{}'' \left\{\sum_{\beta_s}(z^s_\beta - z^{s-1}_\beta)\int_{\partial K^s_\beta} f(\zeta)\,d\zeta +\right.$$

$$\left.+\sum_{\beta_s}{}' \int_{\partial K^s_\beta} f(\zeta)(\zeta - z^s_\beta)\,d\zeta\right\} + \sum_{\beta_0,\dots,\beta_n}{}'' \int_{\partial K^n_\beta} f(\zeta)(\zeta - z^n_\beta)\,d\zeta.$$

Denote by $\omega(\delta)$ the modulus of continuity of $f(z)$ and put

$$\psi(\delta) = \max_{t \leq \delta}[\Omega(t) + \omega(t)]$$

and

$$e^s_\beta = Ce \cap K(z^s_\beta, r^0\delta q^{-s}).$$

From the inequality

$$\left| \int_{\partial K_\beta^s} f(\zeta)(\zeta - z_\beta^s)\, d\zeta \right| \leq \omega(\delta q^{-s}) \cdot 4\delta q^{-s}$$

and Lemma 2 we obtain

$$\left| \int_{\partial K(z,\delta)} f(\zeta)(\zeta - z)\, d\zeta \right| \leq \sum_{s=1}^{n-1} \sum_{\beta_0,\dots,\beta_{s-1}}'' \left\{ \sum_{\beta_s} |z_\beta^s - z_\beta^{s-1}| \gamma(e_\beta^s)\Omega(\delta q^{-s}) + \right.$$

$$+ \sum_{\beta_s}' \omega(\delta q^{-s})\delta q^{-s} \cdot 4\delta q^{-s} + \sum_{\beta_0,\dots,\beta_n}'' \omega(\delta q^{-n}) \cdot 4(\delta q^{-n})^2 \leqslant$$

$$\leqslant \sum_{s=1}^{n-1} \sum_{\beta_0,\dots,\beta_{s-1}}'' \left\{ \delta q^{1-s}\psi(\delta) \sum_{\beta_s} \gamma(e_\beta^s) + 4\delta q^{-s}\psi(\delta) \sum_{\beta_s}' q\gamma(Ce \cap K_\beta^s) \right\}$$

$$+4\delta^2\omega(\delta q^{-n}) \leqslant 4\delta\psi(\delta) \sum_{s=1}^{\infty} q^{1-s} \sum_{\beta_0,\dots,\beta_{s-1}}'' \{c(p)[\gamma(e_\beta^{s-1}) + \gamma(Ce \cap K_\beta^{s-1})]\} \leqslant$$

$$\leqslant 8\delta\psi(\delta) \sum_{s=1}^{\infty} q^{1-s}[c(p)]^s \gamma(e_\beta^0),$$

whence, for $q = [2c(p)+2]$, $\Phi(\delta) = 16\psi(\delta)$, we obtain the assertion of the lemma.

Denote:

$$\beta(e, z, f) = [\gamma(e)2\pi i]^{-1} \int_{\partial e} f(\zeta)(\zeta - z)\, d\xi; \qquad \beta(e, z) = \sup_{f \in A(e,1)} |\beta(e, z, f)|;$$

$$\beta(e) = \min_z \beta(e, z)$$

–the "analytic diameter" of the set e; $O(e)$ is the "center," i.e., a point such that $\beta(e, O(e)) = \beta(e)$.

Lemma 5. If e and $f(z)$ satisfy the conditions of the theorem, then for every square $K(z, \delta)$

$$\left| \int_{\partial K(z,\delta)} f(\zeta)[\zeta - O(Ce \cap K(z, r\delta))]\, d\xi \right| \leqslant$$

$$\leqslant \Lambda(\delta)\gamma(Ce \cap K(z, r\delta))\beta(Ce \cap K(z, r\delta))$$

and $\Lambda(\delta) \to 0$ as $\delta \to 0$.

Proof. Divide $K(z,\delta)$ into equal squares with side β and centers at the points t_i. Put $e(z,\delta) = Ce \cap K(z, r\delta)$. Choose β so that $\beta \leqslant \beta(e(z,\delta)) \leqslant 2\beta$, if $\beta(e(z,\delta)) \leqslant \delta$, and $\beta = \delta$, if $\beta(e(z,\delta)) > \delta$. From the condition of the theorem and Lemmas 4 and 3 we obtain

$$\left| \int_{\partial K(z,\delta)} f(\zeta)[\zeta - O(e(z,\delta))]\, d\xi \right| \leqslant$$

$$\leqslant \sum_i |t_i - O(e(z,\delta))| \gamma(e(t_i,\beta))\Omega(\beta) + \sum_i \gamma(e(t_i,\beta))\beta\Phi(\beta) \leqslant$$

$$\leqslant \sum_i |t_i - O(e(z,\delta))| \gamma(e(t_i,\beta))\Omega(\beta) + c(p)\gamma(e(z,\delta))\beta\Phi(\beta).$$

Let $f_i = f_i(z) \in A(e(t_i,\beta), 1)$ and

$$\gamma(e(t_i,\beta), f_i) = \frac{1}{2}[t_i - O(e(z,\delta))]^{-1}|t_i - O(e(z,\delta))|\gamma(e(t_i,\beta));$$

then $\varphi(z) = \sum_i f_i(z)$ is bounded in modulus by $c(p)$ (see Lemma 2). Consequently,

$$\beta(e(z,\delta)) \geqslant |(c(p))^{-1}\beta(e(z,\delta)),\, O(e(z,\delta))|,$$

$$|\varphi(z)| \geqslant [2\pi\gamma(e(z,\delta))c(p)]^{-1} \left| \int \left(\sum_i [t_i - O(e(z,\delta))] f_i(\zeta) + (\zeta - t_i) f_i(\zeta) \right) d\xi \right| \geqslant$$

$$\geqslant [2\pi\gamma(e(z,\delta))c(p)]^{-1} \left\{ \sum_i \frac{1}{2}|t_i - O(e(z,\delta))|\gamma(e(t_i,\beta)) - m_1(r)\beta\gamma(e(z,\delta)) \right\},$$

where $m_1(r)$ is a constant.

Thus,

$$\left| \int_{\partial K(z,\delta)} f(\zeta)[\zeta - O(e(z,\delta))]\, d\zeta \right| \leq$$

$$\leq \sum_i |t_i - O(e(z,\delta))|\,\gamma(e(t_i,\beta))\Omega(\beta) + c(p)\gamma(e(z,\delta))\beta\Phi(\beta) \leq$$

$$\leq \gamma(e(z,\delta))\beta(e(z,\delta))m_2(r)[\Omega(\beta) + \Phi(\beta)].$$

The lemma is proved.

Lemma 6. If the complex numbers γ and β are such that $|\gamma| \leq \gamma(e)$ and $|\beta| \leq \beta(e)$, then there exists a function $g(z) \in A(e,5)$ for which $\gamma(e,g) = \gamma$ and $\beta(e, O(e), g) = \beta$.

Proof. For every function $f(z)$,

$$\beta(e,\zeta,f) = \beta(e,O(e),f) + \frac{\gamma(e,f)}{\gamma(e)}(\zeta - O(e)).$$

Consequently, for $\varphi = \varphi(z,e)$, when

$$z_0 = O(e) - \beta(e,O(e),\varphi)$$

we have $\beta(e,z_0,\varphi) = 0$.

From the definition of $O(e)$ it follows that for some function $f_1 \in A(e,1)$,

$$\beta(e,z_0,f_1) = \beta(e).$$

Choose ε so that for

$$f_2 = (f_1 + \varepsilon\varphi) \in A(e,2)$$

one has $\beta(e,O(e),f_2) = \beta(e)$ and $\gamma(e,f_2) = 0$. Since

$$\beta(e,O(e),\varphi) = O(e) - z_0,$$

we have $|O(e) - z_0| \leq \beta(e)$. Then

$$f = \frac{\gamma}{\gamma(e)}\varphi - \frac{\gamma[O(e) - z_0]}{\gamma(e)\beta(e)}f_2 + \frac{\beta}{\beta(e)}f_2 \in A(e,5)$$

is the desired function.

Proof of the theorem. Let K be a square containing the set e. Divide it into equal squares

$$K_i = K(t_i,\delta), \qquad O_i = O(Ce \cap K(t_i, r\delta)).$$

Let

$$f^* = \frac{1}{2\pi i}\int_{\partial K}\frac{f^*(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{\pi}\int_K \frac{df^*}{d\bar{\zeta}}\,\frac{1}{\zeta - z}\,dS$$

be a smooth function uniformly approximating f on K with accuracy up to ε. By Lemma 6, there exist functions

$$g_i \in A(Ce, 5(\Delta(\delta) + \Omega(\delta))),$$

whose first two coefficients in the expansion in powers of $1/z - O_i$ are equal to the corresponding coefficients of the function

$$f_i^* = \frac{1}{\pi} \int_{K_i} \frac{df^*}{d\bar{\zeta}} \frac{1}{\zeta - z} \, dS.$$

It remains to verify that, if one first chooses sufficiently small δ, and then ε, then the function analytic in a neighborhood of the set e,

$$g(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f^*(\zeta)}{\zeta - z} \, d\zeta + \sum g_i$$

will be sufficiently close to $f(z)$. Approximating this function by a rational fraction, we obtain the assertion of the theorem.

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Note: Figure translations are in progress. See original paper for figures.