EXPANSION IN EIGENFUNCTIONS OF A ONE-DIMENSIONAL DIRAC SYSTEM

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Abstract

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MATHEMATICS

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EXPANSION IN EIGENFUNCTIONS OF A ONE-DIMENSIONAL DIRAC SYSTEM

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1. Let $\mathcal{L}$ denote the matrix operator

$$\mathcal{L} \equiv \begin{pmatrix} p(x) & d/dx \\ -d/dx & r(x) \end{pmatrix},$$

where $p(x)$ and $r(x)$ are real-valued functions defined on the half-line $(0, \infty)$ and summable on every finite interval. Further, let $y(x)$ denote a two-component vector-function

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

Then the equation (λ is a parameter)

$$(\mathcal{L} - \lambda)y = 0$$

is equivalent to the system of two compatible ordinary differential equations of the first order

$$dy_2/dx + p(x)y_1 = \lambda y_1, \tag{1}$$

$$-dy_1/dx + r(x)y_2 = \lambda y_2. \tag{2}$$

The system (1)–(2) is a one-dimensional analogue of the stationary system in the relativistic quantum theory of Dirac (^{1})

$$(W - V + mc^2)X_1 - ic\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)X_4 - ic\frac{\partial X_3}{\partial z} = 0,$$

$$(W - V + mc^2)X_2 - ic\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)X_3 + ic\frac{\partial X_4}{\partial z} = 0,$$

$$(W - V - mc^2)X_3 - ic\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)X_2 - ic\frac{\partial X_1}{\partial z} = 0, \qquad (3)$$

$$(W - V - mc^2)X_4 - ic\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)X_1 + ic\frac{\partial X_2}{\partial z} = 0,$$

where $V = V(x, y, z)$ is the potential function, c is the speed of light, W is the energy, and m is the mass of the particle.

Indeed, if $X_1(x, y, z) = X_4(x, y, z) \equiv 0$, and the functions $V(x, y, z)$, $X_2(x, y, z)$, and $X_3(x, y, z)$ do not depend on x and z, then system (3) takes the form

$$\{W - V(y) + mc^2\}X_2 + c\frac{dX_3}{dx} = 0, \qquad (4)$$

$$\{W - V(y) - mc^2\}X_3 - c\frac{dX_2}{dx} = 0.$$

Let the units of measurement be chosen so that the speed of light $c = 1$. If we put $p(y) = V(y) + m$, $r(y) = V(y) - m$, $\lambda = W$, then systems (1)–(2) and (4) will differ only in notation.

We adjoin to system (1)–(2) the boundary condition

$$y_2(0) - hy_1(0) = 0, \qquad (5)$$

where h is an arbitrary real number.

Consider the eigenvalue problem (1) + (2) + (5).

In the present note, by a method analogous to the method of B. M. Levitan (3), we study the question of expanding an arbitrary vector-function $f(x) = \{f_1(x), f_2(x)\}$ with square integrable norm on the half-line $(0, \infty)$,

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\}\,dx < +\infty,$$

in the eigenfunctions of the problem (1) + (2) + (5).

2. Denote by $\varphi(x, \lambda) = \{\varphi_1(x, \lambda), \varphi_2(x, \lambda)\}$ the solution of the system (1)–(2) satisfying the initial conditions

$$\varphi_1(0,\lambda) = 1, \qquad \varphi_2(0,\lambda) = h. \tag{5'}$$

It is obvious that the vector-function $\varphi(x,\lambda)$ satisfies condition (5).

It is known (2) that, for a given h, to each boundary-value problem (1) + (2) + (5) there corresponds a unique nondecreasing, bounded on every finite interval, left-continuous function $\rho(\lambda)$ $(-\infty < \lambda < \infty)$, generating an isometric mapping of the space of vector-functions $f(x) \subset L^2(0,\infty)$ onto the space $L^2_{\{\rho(\lambda)\}}(-\infty,\infty)$ by the formulas

$$F(\lambda) = \int_0^\infty \{f_1(x)\varphi_1(x,\lambda) + f_2(x)\varphi_2(x,\lambda)\}\,dx, \tag{6}$$

$$f_1(x) = \int_{-\infty}^\infty F(\lambda)\varphi_1(x,\lambda)\,d\rho(\lambda), \tag{7}$$

$$f_2(x) = \int_{-\infty}^\infty F(\lambda)\varphi_2(x,\lambda)\,d\rho(\lambda), \tag{8}$$

where the integrals (6) and (7)–(8) converge respectively in the metrics of the spaces $L^2_{\{\rho(\lambda)\}}(-\infty,\infty)$ and $L^2(0,\infty)$, and Parseval' s equality holds:

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\}\,dx = \int_{-\infty}^\infty F^2(\lambda)\,d\rho(\lambda).$$

Put $(i,k = 1,2)$

$$\Theta_{ik}(x,s;\lambda) = \begin{cases} \displaystyle\int_0^\lambda \varphi_i(x,\lambda)\varphi_k(s,\lambda)\,d\rho(\lambda), & \lambda > 0, \\ \displaystyle -\int_\lambda^0 \varphi_i(x,\lambda)\varphi_k(s,\lambda)\,d\rho(\lambda), & \lambda < 0, \\ 0, & \lambda = 0. \end{cases}$$

The square matrix of second order $\Theta(x,s;\lambda) = \{\Theta_{ik}(x,s;\lambda)\}$ is called the spectral matrix of the boundary-value problem (1) + (2) + (5).

Let the vector-function $f(x) = \{f_1(x), f_2(x)\} \subset L^2(0,\infty)$, i.e.

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\}\,dx < +\infty.$$

Introduce the notation

$$S_1(x,\lambda) = \int_0^\infty \{f_1(s)\Theta_{11}(x,s;\lambda) + f_2(s)\Theta_{12}(x,s;\lambda)\}\,ds, \tag{9}$$

$$S_2(x,\lambda) = \int_0^\infty \{f_1(s)\Theta_{21}(x,s;\lambda) + f_2(s)\Theta_{22}(x,s;\lambda)\}\,ds. \tag{10}$$

By virtue of the definition of the functions $\Theta_{ik}(x,s;\lambda)$ and equality (6), for the functions $S_1(x,\lambda)$ and $S_2(x,\lambda)$ we obtain the expressions

$$S_1(x,\lambda) = \int_0^\lambda F(\lambda)\varphi_1(x,\lambda)\,d\rho(\lambda),$$

$$S_2(x,\lambda) = \int_0^\lambda F(\lambda)\varphi_2(x,\lambda)\,d\rho(\lambda).$$

The functions $S_1(x,\lambda)$ and $S_2(x,\lambda)$ are segments of the expansions, respectively, of the functions $f_1(x)$ and $f_2(x)$ in the Fourier integral with respect to the eigenfunctions of problem (1) + (2) + (5).

Consider problem (1) + (2) + (5′) for $p(x) = r(x) = 0$ and $h = 0$. Then $\varphi_1(x,\lambda) = \cos\lambda x$, $\varphi_2(x,\lambda) = \sin\lambda x$. Therefore, in the case under consideration the spectral matrix, which we denote by $\Theta^*(x,s;\lambda)$, is determined by the formulas

$$\Theta^*_{11}(x,s;\lambda) = \frac{1}{\pi}\int_0^\lambda \cos\lambda x\cos\lambda s\,d\lambda,$$

$$\Theta^*_{12}(x,s;\lambda) = \frac{1}{\pi}\int_0^\lambda \cos\lambda x\sin\lambda s\,d\lambda,$$

$$\Theta^*_{21}(x,s;\lambda) = \frac{1}{\pi}\int_0^\lambda \sin\lambda x\cos\lambda s\,d\lambda,$$

$$\Theta^*_{22}(x,s;\lambda) = \frac{1}{\pi}\int_0^\lambda \sin\lambda x\sin\lambda s\,d\lambda.$$

Consequently, the functions (analogously to the preceding)

$$S^*_1(x,\lambda) = \int_0^\infty \{f_1(s)\Theta^*_{11}(x,s;\lambda) + f_2(s)\Theta^*_{12}(x,s;\lambda)\}\,ds,$$

$$S_2^*(x,\lambda) = \int_0^\infty \{f_1(s)\Theta_{21}^*(x,s;\lambda) + f_2(s)\Theta_{22}^*(x,s;\lambda)\}\,ds$$

are segments of the expansions of the functions $f_1(s)$ and $f_2(s)$ in the ordinary Fourier integral.

Using the asymptotic estimates for the spectral matrix

$$\Theta(x,s;\lambda) = \{\Theta_{ik}(x,s;\lambda)\} \quad (i,k = 1,2),$$

obtained in note (^{5}), on the basis of the definition of the functions $S_1(x,\lambda)$ and $S_2(x,\lambda)$, i.e., equalities (9) and (10), the following is proved.

Lemma 1. If the coefficients $p(x)$ and $r(x)$ are summable on each finite interval, then for every fixed x and as $|a| \to \infty$ the asymptotic estimates

$$\bigvee_a^{a+1} \{S_i(x,\lambda)\} = o(1), \quad (i = 1,2). \tag{11}$$

hold. The asymptotic estimates (11) hold uniformly in each finite interval of variation of x.

From the definitions of the spectral matrix

$$\Theta^*(x,s;\lambda) = \{\Theta_{ik}^*(x,s;\lambda)\}$$

and the functions $S_1^*(x,\lambda)$ and $S_2^*(x,\lambda)$ it follows directly that

Lemma 2. As $|a| \to \infty$, uniformly on the half-line $(0,\infty)$, the asymptotic estimate

$$\bigvee_a^{a+1} \{S_i^*(x,\lambda)\} = o(1) \quad (i = 1,2)$$

holds.

Lemmas 1 and 2 make it possible to apply Tauberian theorem for Fourier integrals of B. M. Levitan (^{4}).

As a result, one can obtain the following theorem:

Theorem 1 (on equiconvergence). Let $f(x) = \{f_1(x), f_2(x)\} \subset L^2(0,\infty)$. If the coefficients $p(x)$ and $r(x)$ are summable on every finite interval, then for each fixed x the equalities

$$\lim_{\lambda\to\infty} \{[S_1(x,\lambda) - S_1(x,-\lambda)] - [S_1^*(x,\lambda) - S_1^*(x,-\lambda)]\} = 0, \tag{12}$$

$$\lim_{\lambda\to\infty}\{[S_2(x,\lambda)-S_2(x,-\lambda)]-[S_2^*(x,\lambda)-S_2^*(x,-\lambda)]\}=0. \tag{13}$$

hold.

The equalities (12) and (13) hold uniformly on every finite interval of variation of x.

Since, by virtue of the definitions of the functions $S_1(x,\lambda)$ and $S_2^*(x,\lambda)$,

$$S_1^*(x,\lambda)-S_1^*(x,-\lambda)=\frac{1}{\pi}\int_{-\lambda}^{\lambda}\left\{\int_0^{\infty}f_1(s)\cos\lambda s\,ds\right\}\cos\lambda x\,d\lambda,$$

$$S_2^*(x,\lambda)-S_2^*(x,-\lambda)=\frac{1}{\pi}\int_{-\lambda}^{\lambda}\left\{\int_0^{\infty}f_2(s)\sin\lambda s\,ds\right\}\sin\lambda x\,d\lambda,$$

the equalities (12) and (13) have the form

$$\lim_{\lambda\to\infty}\left\{S_1(x,\lambda)-S_1(x,-\lambda)-\frac{1}{\pi}\int_{-\lambda}^{\lambda}\left[\int_0^{\infty}f_1(s)\cos\lambda s\,ds\right]\cos\lambda x\,d\lambda\right\}=0, \tag{14}$$

$$\lim_{\lambda\to\infty}\left\{S_2(x,\lambda)-S_2(x,-\lambda)-\frac{1}{\pi}\int_{-\lambda}^{\lambda}\left[\int_0^{\infty}f_2(s)\sin\lambda s\,ds\right]\sin\lambda x\,d\lambda\right\}=0. \tag{15}$$

The equalities (14) and (15) mean that, uniformly on every interval of the half-line $(0,\infty)$, the difference between the expansion of an arbitrary vector-function $f(x)=\{f_1(x),f_2(x)\}\subset L^2(0,\infty)$ in the generalized Fourier integral with respect to the eigenfunctions of the one-dimensional Dirac operator and the expansion in the ordinary Fourier integral tends to zero.

Theorem 1 gives the final solution of the question of convergence of the expansion in eigenfunctions of the one-dimensional Dirac operator for a two-component vector-function with integrable square.

In particular, from Theorem 1 it follows that

Theorem 2 (on convergence). If the coefficients $p(x)$ and $r(x)$ are summable functions on every finite interval and the vector-function $f(x)=\{f_1(x),f_2(x)\}$ belongs to the class $L^2(0,\infty)$, then at every point where the local conditions of expandability of the vector-function $f(x)$ into the ordinary Fourier integral are satisfied, the equalities

$$\lim_{\lambda\to\infty}\{S_1(x,\lambda)-S_1(x,-\lambda)\}=f_1(x),$$

$$\lim_{\lambda\to\infty}\{S_2(x,\lambda)-S_2(x,-\lambda)\}=f_2(x),$$

hold, i.e. the expansion of the vector-function $f(x)=\{f_1(x),f_2(x)\}$ in the generalized Fourier integral with respect to the eigenfunctions of the one-dimensional Dirac operator tends to the value of the vector-function.

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Note: Figure translations are in progress. See original paper for figures.

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