ON SOME DIFFERENTIAL-TOPOLOGICAL INVARIANTS OF NON-SIMPLY CONNECTED MANIFOLDS

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Abstract

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In differential topology the following problem often arises (we shall call it **Problem I**): in what case does an open manifold with two ends W^m, $m \geqslant 6$, have the structure of a smooth direct product by a line, i.e., when is W^m diffeomorphic to a manifold $V^{m-1} \times R$, where the manifold V^{m-1} is smooth. For example, we shall be interested in a manifold W^m of the homotopy type of a finite complex, on which acts a transformation $T : W \to W$ that is a PL-isomorphism. The fundamental group $\pi = \pi_1(W)$ is assumed to be finite.

Also adjacent to this question are: **Problem II**, on the construction of the boundary of an open manifold; **Problem III**, on smooth fibrations of manifolds over a circle, somewhat more complicated than Problem II.

For the simply connected case, Problem I was considered by Browder ([4]); S. P. Novikov ([2]) studied the case when $\pi_1(W)$ is free abelian. Problems II and III were solved for the simply connected case, respectively, by Browder, Levine, and Livesay ([6]) and by Browder and Levine ([5]). Siebenmann considered the problem of constructing the boundary for the non-simply connected case, but his results have not yet been published, and the author has no precise information about the formulation of his theorems. Wall ([8]) introduced an invariant that is an obstruction to the finiteness of a CW-complex. The situation considered by him remotely resembles ours, but the relations between them are not clear.

We shall restrict ourselves to Problem I; the remaining cases are analogous. As is known (([2]), Appendix 2), an obstruction to introducing the structure of a direct product arises; it is a projective, but not free, module over $Z[\pi]$, i.e., an element of the reduced Grothendieck group $\widetilde{K}^0(\pi)$. In the group $\widetilde{K}^0(\pi)$ there is a canonical involution $a \to a^*$, generated by the functor $\mathrm{Hom}(a, Z)$. In ([2]) Grothendieck groups $D(\pi)$ and $E(\pi)$ were defined for projective modules with a unimodular scalar product, symmetric for $D(\pi)$ and skew-symmetric for $E(\pi)$. There too homomorphisms $\lambda_\pm : \widetilde{K}^0(\pi) \to D(\pi), E(\pi)$ were defined by the formula

$$\lambda_{\pm}(a) = a + a^*.$$

Definition 1. Denote by $G_1^{\pm}$ the kernels of the homomorphisms $\lambda_{\pm}$.

Definition 2. Denote by G_0 the subgroup in $\widetilde{K}^0(\pi)$ generated by elements of the form $a + a^*$.

S. P. Novikov ([2]) proved that in the manifold W^{n+1} one can construct a submanifold V^n of codimension 1 such that the embedding $V^n \subset W^{n+1}$ will induce an isomorphism of $Z[\pi]$-modules of homology on universal coverings in dimensions $< [n/2]$, and in dimension $[n/2]$ an epimorphism. The kernel of this epimorphism decomposes into a left and a right part α and β, $\alpha, \beta \in \widetilde{K}^0(\pi)$. If $\dim V = 2n$, then both modules α and β are projective. If $\dim V = 2n+1$, then, according to ([2]), one can arrange that the right kernel β in dimension n be equal to 0; then the left kernel α in dimension n is projective and contains no torsions.

Keeping the notation, we introduce

Definition 3. For a given submanifold $V \subset W$ and a fixed ordering of the ends of the manifold W, set

$$\Delta(W) = \alpha.$$

The properties of mappings of degree +1 and simple arguments with the Grothendieck functor $\widetilde{K}^0(\pi)$ make it possible to prove Theorem 1.

Theorem 1. *Let there be two closed oriented manifolds V, V_1 with finite fundamental group $\pi_1(V) \cong \pi(V_1)$, and a mapping of degree +1, $f : V \to V_1$, inducing an isomorphism of homotopy groups in dimensions $< [n/2]$. Then the following fact holds:*

(1) if $\dim V = 2n$, then the kernel in the homology of the universal covering in dimension n is a stably free $Z[\pi]$-module;

(2) if $\dim V = 2n+1$ and the kernel Φ in the homology of the universal covering in dimension n is projective, then $\Phi = \Phi^*$.

Theorem 2. *(1) For a fixed ordering of the ends of a manifold W there is the duality relation*

$$\Delta^* = (-1)^{\dim W} \Delta.$$

(2) $\Delta(W) = 0$ if and only if W admits the structure of a smooth direct product.

Theorem 2 is proved by arguments close to those used in the proof of Theorem 1. Item (1) recalls the duality for Whitehead torsion; however, this is essential only in the case when the boundaries of the h-cobordism are diffeomorphic ([11]).

Definition 4. (1) Denote by A^+ and A^- the subgroups in $\widetilde{K}^0(\pi)$

$$A^+ = \{a \in \widetilde{K}^0(\pi) \mid a + a^* = 0\}, \qquad A^- = \{a \in \widetilde{K}^0(\pi) \mid a - a^* = 0\}.$$

(2) Denote by $\mathcal{A}^+$ and $\mathcal{A}^-$ the quotient groups of the Whitehead group

$$\widetilde{K}'(\pi) = \widetilde{K}'(\pi)/\operatorname{im}(\pi)$$

by the subgroups

$$\mathcal{B}^\pm = \{x \mid x = \tau \pm \tau,\ \tau \in \widetilde{K}'(\pi)\}$$

$$\mathcal{A}^+ = \widetilde{K}'(\pi)/\mathcal{B}^+, \qquad \mathcal{A}^- = \widetilde{K}'(\pi)/\mathcal{B}^-.$$

From item (2) of Theorem 2 it is clear that the group A^+ (respectively A^-) gives an obstruction to the diffeomorphism $W = V \times R$, if $\dim W$ is odd (respectively even). The group $\mathcal{A}^+$ (respectively $\mathcal{A}^-$) gives an obstruction to the uniqueness of a manifold V such that $W = V \times R$, if $\dim W$ is odd (respectively even) ([11]).

We proceed to the realization of the invariant $\Delta(W)$.

Definition 5. For a given smooth closed manifold V, denote by $H(V)$ the subset in $\widetilde{K}^0(\pi)$ of elements Δ such that for every $\Delta \in H(V)$ there exists a W of the homotopy type V ($\pi_1(V) = \pi$) with value of the invariant $\Delta(W) = \Delta$. By $H(\pi)$ denote the union

$$H(\pi) = \bigcup_{(V)} H(V).$$

Theorem 3. *(1) $\Delta(W)$ is an invariant of diffeomorphisms and the set $H(\pi)$ is a subgroup in $\widetilde{K}^0(\pi)$; (2) for a given manifold V the following inclusions hold: (a) $H(V) \subset G_1^+ \subset H(\pi)$, if $\dim V = 4n$; (b) $H(V) \subset G_1^- \subset H(\pi)$, if $\dim V = 4n+2$; (c) $H(V) \subset G_0 \subset H(\pi)$, if $\dim V = 2n+1$.*

At the basis of our construction of a realization lies an observation due to Eilenberg—the construction, for any projective module M, of a free module of infinite rank F_1 such that $M + F_1$ is a free module F_2. Indeed, if the module N is such that $M + N$ is free, then F_1 is the infinite direct sum $N + M + N + M + \cdots$, since

$$M + F_1 = (M + N) + (M + N) + \cdots.$$

In our case the module F_1 will be generated by handles of index n, attached along the trivial embedding of the boundary, and the module F_2 by handles of index $n+1$.

We restrict ourselves to the case $\dim W = 4n+1$. The other two are analogous. For an arbitrary element $\gamma \in G_1^+$ we have $\Phi = \gamma + \gamma^* = F + F^*$, F a free module. Take a smooth manifold V of arbitrary homotopy type with fundamental group $\pi_1(V) \cong \pi$, and form the connected sum

$$V_0 = V \# (S_1^{2n} \times S_1^{2n}) \# \ldots \# (S_N^{2n} \times S_N^{2n}),$$

where N is large. For the homology modules of the universal coverings we have the direct decomposition

$$H_*(\hat{V}_0) = H_*(\hat{V}) + \Phi, \qquad \Phi = F + F^*,$$

where F is a free module.

Each element of the module γ is realized by an embedded sphere, since $\gamma \cap \gamma^* = 0$ ([2]). The module γ will be a left kernel, and the module γ^* a right one. Multiply V_0 by the interval $I = [0, 1]$, $\omega_0 = V_0 \times I$. We shall kill the module γ on the left, and the module γ^* on the right. Attaching to $V_0 \times 1$ handles of index n, along trivial embeddings of the boundary, we can realize a free module in n-dimensional homologies $F_1 \cong \Phi$. Regroup the summands in the sum $\gamma + F_1 = F_2 + \gamma$, where $F_2 \cong \Phi$. The module F_2 is realized by embedded spheres ([9]). Therefore we can kill F_2 by attaching handles of index $n+1$. When handles of index $n+1$ are attached, the boundary V_0' is subjected to Morse surgeries. In homology there remains again the module γ. The construction is easily carried out so that the rebuilt manifold V_0'' is diffeomorphic to the original V_0. One could repeat this process, thus going off to infinity, which is precisely what corresponds to Eilenberg' s remark. But in view of the fact that we obtain a boundary diffeomorphic to the original one, it is enough to take

$$\ldots \cup \omega''_{-2} \cup \omega''_{-1} \cup \omega''_0 \cup \omega''_1 \cup \omega''_2 \ldots = W,$$

where we glue adjacent films by a diffeomorphism of their boundaries. It is easy to see that $\Delta(W) = \gamma$ and that there exists a transformation $T : W \to W$, where $T(\omega''_i) = \omega''_{i+1}$, and W has the homotopy type of V.

Let us pass to the application of the results of Theorems 1–3 to concrete examples on the basis of algebraic information obtained in the theory of class fields. If $\pi = Z_p$, p prime, then $\widetilde{K}^0(\pi)$ is isomorphic to the ideal class group of the field of division of the circle into p parts ([7]).

Example. We now consider an example illustrating our results. Take $\pi = Z_{23}$. It is known that $\widetilde{K}^0(Z_{23}) \supset Z_3$ ([1,7]). Consider the product $L \times S^{2k+1}$, where L is a lens space of large dimension with fundamental group $\pi_1(L) = Z_{23}$. From Theorem 3 it follows that we can construct a manifold $W \sim L \times S^{2k+1}$, for which $\Delta(W) = \alpha \in \widetilde{K}^0(Z_{23})$, $\alpha \neq 0$.

Corollary 2. There exist closed smooth manifolds $\overline{W}$ with fundamental group $Z + Z_{23}$, whose homotopy groups are finitely generated, which are not bundles with base the circle (for $\pi_1 = Z$ this cannot occur ([5])).

Corollary 3. There exist open manifolds W, not diffeomorphic to $V \times R$ and not an open part of a manifold with boundary, which are obtained by a covering $W \to \overline{W}$ with group of deck transformations Z from the manifolds of Corollary 2.

In conclusion we note that everything set forth is also true for PL-manifolds, since the corresponding variant of Morse surgeries reduces to the smooth case ([3], Appendix 2).

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Note: Figure translations are in progress. See original paper for figures.