ASYMPTOTIC BEHAVIOR OF THE SPECTRAL MATRIX OF A ONE-DIMENSIONAL DIRAC SYSTEM

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Abstract

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ASYMPTOTIC BEHAVIOR OF THE SPECTRAL MATRIX OF A ONE-DIMENSIONAL DIRAC SYSTEM

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1. In relativistic Dirac quantum theory the wave function Ψ satisfies an equation of the form

$$(p_0 + Vc^{-1} + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta)\Psi = 0, \tag{1}$$

where $V = V(x, y, z)$ is the potential function, c is the speed of light,

$$p_0 = i\hbar c^{-1}\frac{\partial}{\partial t}, \qquad p_x = -i\hbar\frac{\partial}{\partial x}, \qquad p_y = -i\hbar\frac{\partial}{\partial y}, \qquad p_z = -i\hbar\frac{\partial}{\partial z},$$

$\hbar$ is Planck' s constant, and the operators $\alpha_x, \alpha_y, \alpha_z$ and β satisfy the conditions

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1, \qquad \alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_x\alpha_z + \alpha_z\alpha_x = \alpha_y\alpha_z + \alpha_z\alpha_y = 0,$$

$$\beta^2 = m^2c^2, \qquad \alpha_x\beta + \beta\alpha_x = \alpha_y\beta + \beta\alpha_y = \alpha_z\beta + \beta\alpha_z = 0, \tag{2}$$

and, finally, m is the mass of the particle. These formulas follow from formula (19), § 53.1 of [1], if in the latter formula we set $A = 0$ and $e\Phi = V$. Let the operators $\alpha_x, \alpha_y, \alpha_z$, and β be square 4×4 matrices. Then the wave function Ψ will be a 4×1 matrix, i.e., a four-component function

$$\Psi = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}, \qquad \Psi_k = \Psi_k(x, y, z; t) \quad (k = 1, 2, 3, 4),$$

and therefore equation (1) is equivalent to a system of four partial differential equations for the functions $\Psi_k = \Psi_k(x, y, z; t)$ $(k = 1, 2, 3, 4)$. If we put $\Psi = e^{-iWt/\hbar}X$, where W is the energy, and the four-component function X depends only on x, y, and z, then equation (1) takes the form

$$\{(W-V)c^{-1}+\alpha_{xp}x+\alpha_{yp}y+\alpha_{zp}z+\beta\}X=0. \tag{3}$$

Here W plays the role of the eigenvalue parameter.

2. We shall consider the one-dimensional case of system (3), i.e., assume that the functions V and X depend only on x. Then from system (3) we obtain

$$\{(W-V)c^{-1}+\alpha p_x+\beta\}X=0, \tag{4}$$

where

$$p_x=-i\hbar\frac{d}{dx}, \qquad \alpha^2=1, \qquad \beta^2=m^2c^2, \qquad \alpha\beta+\beta\alpha=0. \tag{5}$$

In this case α and β are 2×2 matrices, while the function X is now two-component and may be regarded as a 2×1 matrix. Therefore equation (4) is equivalent to two simultaneous ordinary differential equations of first order for the components $X_1(x)$ and $X_2(x)$ of the function $X(x)$.

Let

$$\alpha=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \frac{\beta}{mc}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad X(x)=\begin{pmatrix}X_1(x)\\ X_2(x)\end{pmatrix}.$$

Then conditions (5) are satisfied, and therefore from equation (4) it follows that

$$\begin{aligned}\{W-V(x)\}c^{-1}X_1-\hbar X_2'+mcX_1&=0,\\ \{W-V(x)\}c^{-1}X_2+\hbar X_1'-mcX_2&=0.\end{aligned} \tag{6}$$

Suppose that the units of measurement are chosen so that the speed of light c and Planck’ s constant $\hbar$ are equal to 1. Then, also putting

$$p(x)=V(x)-m, \quad r(x)=V(x)+m, \quad \lambda=W,$$

we can write system (6) in the form

$$\begin{aligned}dX_2/dx+p(x)X_1&=\lambda X_1,\\ -dX_1/dx+r(x)X_2&=\lambda X_2.\end{aligned} \tag{7}$$

Let us adjoin to system (7) the boundary condition

$$X_2(0)-hX_1(0)=0, \tag{8}$$

where h is an arbitrary real number.

Denote by $\varphi(x,\lambda)=\{\varphi_1(x,\lambda),\varphi_2(x,\lambda)\}$ the solution of system (7) satisfying the initial conditions

$$\varphi_1(0,\lambda)=1,\quad \varphi_2(0,\lambda)=h. \tag{8'}$$

It is obvious that the vector-function $\varphi(x,\lambda)=\{\varphi_1(x,\lambda),\varphi_2(x,\lambda)\}$ satisfies the boundary condition (8).

It is known ($^{2-3}$) that for a given h, to each boundary-value problem (7)–(8) there corresponds a unique* nondecreasing, bounded on every finite interval, left-continuous function $\rho(\lambda)$ $(-\infty<\lambda<\infty)$, which generates an isometric mapping of the space of vector-functions $f(x)=\{f_1(x),f_2(x)\}\subset\mathcal{L}^2(0,\infty)$ square-integrable on the half-line $(0,\infty)$:

$$\int_0^\infty\{f_1^2(x)+f_2^2(x)\}\,dx<+\infty,$$

onto the space $\mathcal{L}_{\{\rho(\lambda)\}}(-\infty,\infty)$ by the formulas

$$F(\lambda)=\int_0^\infty\{f_1(x)\varphi_1(x,\lambda)+f_2(x)\varphi_2(x,\lambda)\}\,dx, \tag{9}$$

$$f_1(x)=\int_{-\infty}^\infty F(\lambda)\varphi_1(x,\lambda)\,d\rho(\lambda), \tag{10}$$

$$f_2(x)=\int_{-\infty}^\infty F(\lambda)\varphi_2(x,\lambda)\,d\rho(\lambda), \tag{11}$$

where the integrals (9) and (10)–(11) converge in the metrics of the spaces $\mathcal{L}^2_{\{\rho(\lambda)\}}(-\infty,\infty)$ and $\mathcal{L}^2(0,\infty)$, respectively, and Parseval' s equality holds:

$$\int_0^\infty\{f_1^2(x)+f_2^2(x)\}\,dx=\int_{-\infty}^\infty F^2(\lambda)\,d\rho(\lambda).$$

3. Let us introduce the following terminology: the function $\rho(\lambda)$ will be called the **spectral function** of problem (7)–(8); the matrix of second order $\Theta(x,s;\lambda)=\{\Theta_{ik}(x,s;\lambda)\}$ $(i,k=1,2)$, where the functions $\Theta_{ik}(x,s;\lambda)$

* The uniqueness of the function $\rho(\lambda)$ was proved by B. M. Levitan.

are defined by the formulas

$$\Theta_{ik}(x,s;\lambda) = \begin{cases} \displaystyle\int_0^{\lambda} \varphi_i(x,\lambda)\varphi_k(s,\lambda)\,d\rho(\lambda), & \lambda > 0, \\ \displaystyle -\int_{\lambda}^{0} \varphi_i(x,\lambda)\varphi_k(s,\lambda)\,d\rho(\lambda), & \lambda < 0, \\ 0, & \lambda = 0, \end{cases} \tag{12}$$

we shall call the **spectral matrix** of problem (7)–(8).

In the present note, by means of a method analogous to B. M. Levitan' s method (^{4}), the asymptotic behavior of the spectral matrix $\Theta(x,s;\lambda)$ as $|\lambda| \to \infty$ is studied. We shall compare the spectral matrix $\Theta(x,s;\lambda)$ with the spectral matrix $\Theta^*(x,s;\lambda)$ of a simpler problem, namely, problem (7)–(8) with $p(x) = r(x) \equiv 0$ and $h = 0$. In this case $\varphi_1(x,\lambda) = \cos\lambda x$, $\varphi_2(x,\lambda) = \sin\lambda x$, and therefore the matrix $\Theta^*(x,s;\lambda)$ is defined by the formulas

$$\Theta^*_{11}(x,s;\lambda) = \frac{1}{\pi}\int_0^{\lambda} \cos\lambda x\cos\lambda s\,d\lambda = \frac{1}{2\pi}\left\{\frac{\sin\lambda(x-s)}{x-s} + \frac{\sin\lambda(x+s)}{x+s}\right\},$$

$$\Theta^*_{12}(x,s;\lambda) = \frac{1}{\pi}\int_0^{\lambda} \cos\lambda x\sin\lambda s\,d\lambda =$$

$$= -\frac{1}{2\pi}\left\{\frac{\cos\lambda(x-s)}{x-s} + \frac{\cos\lambda(x+s)}{x+s}\right\} + \frac{1}{\pi}\frac{x}{x^2-s^2},$$

$$\Theta^*_{21}(x,s;\lambda) = \frac{1}{\pi}\int_0^{\lambda} \sin\lambda x\cos\lambda s\,d\lambda =$$

$$= \frac{1}{2\pi}\left\{\frac{\cos\lambda(x-s)}{x-s} - \frac{\cos\lambda(x+s)}{x+s}\right\} - \frac{1}{\pi}\frac{x}{x^2-s^2},$$

$$\Theta^*_{22}(x,s;\lambda) = \frac{1}{\pi}\int_0^{\lambda} \sin\lambda x\sin\lambda s\,d\lambda = \frac{1}{2\pi}\left\{\frac{\sin\lambda(x-s)}{x-s} - \frac{\sin\lambda(x+s)}{x+s}\right\}.$$

Lemma. If the coefficients $p(x)$ and $r(x)$ are summable on every finite interval and (x_0,x_1) is an arbitrary finite interval, then there exists a constant $C = C(x_0,x_1)$ such that, for all values x and s belonging to the interval (x_0,x_1), and for all a,

$$\bigvee_{a}^{a+1}\{\Theta_{ik}(x,s;\lambda)\} < C \qquad (i,k = 1,2). \tag{13}$$

Estimate (13) for the matrix $\Theta^*(x,s;\lambda)$ is obtained directly, if one takes into account the explicit form of the functions $\Theta^*_{ik}(x,s;\lambda)$ $(i,k=1,2)$.

Estimate (13) makes it possible to apply Tauber' s theorem for Fourier integrals of V. A. Marchenko (^{5}). As a result, the following theorem can be proved:

Theorem 1. If the coefficients $p(x)$ and $r(x)$ satisfy Dini' s condition, then for every fixed x and s the following asymptotic formulas hold

$$\lim_{\lambda\to\infty}\left\{\Theta_{11}(x,s;\lambda)-\Theta_{11}(x,s;-\lambda)-\frac{1}{\pi}\left[\frac{\sin\lambda(x-s)}{x-s}+\frac{\sin\lambda(x+s)}{x+s}\right]\right.$$
$$\left.+\frac{4}{\pi}\,\frac{\sin\lambda(x-s)}{x-s}\,\sin^2\left[\frac{1}{4}\int_s^x\{p(\tau)+r(\tau)\}\,d\tau\right]\right\}=0, \tag{14}$$

$$\lim_{\lambda\to\infty}\left\{\Theta_{12}(x,s;\lambda)-\Theta_{12}(x,s;-\lambda)-\frac{2}{\pi}\frac{\sin\lambda(x-s)}{x-s}\sin\left[\frac{1}{2}\int_s^x\{p(\tau)+r(\tau)\}\right]\right\}=0, \tag{15}$$

$$\lim_{\lambda\to\infty}\left\{\Theta_{21}(x,s;\lambda)-\Theta_{21}(x,s;-\lambda)+\frac{2}{\pi}\frac{\sin\lambda(x-s)}{x-s}\sin\left[\frac{1}{2}\int_s^x\{p(\tau)+r(\tau)\}\,d\tau\right]\right\}=0, \tag{16}$$

$$\lim_{\lambda\to\infty}\left\{\Theta_{22}(x,s;\lambda)-\Theta_{22}(x,s;\lambda)-\frac{1}{\pi}\left[\frac{\sin\lambda(x-s)}{x-s}-\frac{\sin\lambda(x+s)}{x+s}\right]+\frac{4}{\pi}\frac{\sin\lambda(x-s)}{x-s}\sin^2\left[\frac{1}{4}\int_s^x\{p(\tau)+r(\tau)\}\,d \tag{17}$$

The asymptotic formulas (14)–(17) hold uniformly in each finite domain of variation of the variables x and s.

From Theorem 1, more precisely from formula (14), for $x=s=0$ it follows (by virtue of (8′) and (12))

Theorem 2. If the coefficients $p(x)$ and $r(x)$ satisfy Dini' s condition, then the asymptotic formula

$$\lim_{\lambda\to\infty}\left\{\rho(\lambda)-\rho(-\lambda)-\frac{2}{\pi}\lambda\right\}=0$$

holds.

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Note: Figure translations are in progress. See original paper for figures.