STABILITY REGIONS OF COMPLEX SELF-ADJOINT SYSTEMS OF DIFFERENTIAL EQUATIONS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.941.92

MATHEMATICS

V. B. LIDSKII, P. A. FROLOV

STABILITY REGIONS OF COMPLEX SELF-ADJOINT SYSTEMS OF DIFFERENTIAL EQUATIONS

(Presented by Academician A. A. Dorodnitsyn on 21 I 1966)

We consider a complex linear system of n differential equations

$$Q(t)\frac{dy}{dt} - \left\{S(t) - \frac{1}{2}\frac{d}{dt}Q(t)\right\} y = 0, \tag{1}$$

where $Q(t)$ is a nonsingular skew-Hermitian matrix: $Q^*(t) = -Q(t)$; $S(t)$ is a Hermitian matrix: $S^*(t) = S(t)$; y is an unknown column vector.* It is assumed that the matrices $Q(t)$ and $S(t)$ have periods equal to ω; all coefficients of system (1) are regarded as piecewise continuous.

A system (1) is called **strongly stable** if all its solutions are bounded as $t \to \infty$ and all systems of the form (1) whose coefficients deviate sufficiently little, in absolute value, from the coefficients of the given system possess this property.

The set $\mathfrak{M}$ of all strongly stable systems is thus open and, consequently, decomposes into no more than a countable number of connected components. Each connected component of the set $\mathfrak{M}$ is called a **stability region**. In the present note the topological structure of $\mathfrak{M}$ is studied. In the case where $Q(t)$ in (1) is a constant matrix, this problem was investigated in the works ([1,2]). An **infinite** number of stability regions was discovered and their characteristics were found.** Below we establish that, in the case of the general system (1), the set $\mathfrak{M}$ consists only of a finite number of components.***

Let $i\varkappa_1(t), i\varkappa_2(t), \ldots, i\varkappa_n(t)$ be the eigenvalues of the matrix $Q(t)$; since $Q(t)$ is nonsingular and continuous, all functions $\varkappa_s(t)$ preserve their sign. Suppose that among them p are positive and q are negative. Without loss of generality, in what follows we shall everywhere assume that

$$-iQ(0) = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}. \tag{2}$$

Fig. 1

Figure 1: Fig. 1

We shall denote the matrix on the right-hand side of (2) by J. Since the multiplier theory of M. G. Krein (see ([4])) carries over to systems (1), one can, by analogy with the real case (see ([3])), establish the following proposition:

* The left-hand side of (1) is a general formally self-adjoint differential operator of first order in the space of c

** Unfortunately, inaccuracies were made in the determination of the characteristics in both works.**

*** An analogous fact was established earlier by the authors of the present article in the real case (see ([3])); however, the topological characteristics and the total number of connected components in the complex case turn out to be different.**

For a nonsingular piecewise differentiable matrix

$$Y(t), \qquad 0 \leqslant t \leqslant \omega, \tag{3}$$

to be the matricant of a strongly stable system (1), it is necessary and sufficient that the matrix $Y(\omega)$ be a J-unitary matrix of stable type.

The latter means that $Y^*(\omega)JP(\omega) = J$, all eigenvalues of $Y(\omega)$ (multipliers) lie on the unit circle, and among them there are no multiple ones of different kinds. Relying on the fact indicated above, we proceed according to the same plan as that adopted in [5], and reduce the problem to the investigation of the structure of the set of curves (3).

Fig. 1

We shall agree to denote multipliers of the 1st kind by the symbol $\oplus$, and of the 2nd kind by the symbol $\ominus$.

Consider all possible permutations of p symbols $\oplus$ and q symbols $\ominus$ on the unit circle. Let $N_{p,q}$ be the total number of the indicated permutations*.

It can be shown (see [5], Lemma 2, and [2], Theorem 6) that the set Ω of all J-unitary matrices of stable type consists of $N_{p,q}$ open connected components Ω_P, each of which is characterized by a permutation P formed by multipliers of the 1st and 2nd kinds on the circle. Therefore, if the ends of two curves $Y^{(1)}(t)$ and $Y^{(2)}(t)$ belong to one and the same component Ω_P of matrices of stable type, then they can be joined without violating the conditions of strong stability. This is equivalent to constructing a curve $W(\tau)$, $0 \leqslant \tau \leqslant 1$, which

Fig. 2

Figure 2: Fig. 2

joins $Y^{(1)}(\omega)$ with $Y^{(2)}(\omega)$ (see Fig. 1) and lies entirely in the component Ω_P, and to the subsequent change of the parameter. After this, the question of a continuous deformation of two curves $Y^{(1)}(t)$ and $Y^{(2)}(t)$ with fixed ends in the group of all nonsingular matrices is solved simply. For such a deformation to exist it is necessary and sufficient that

$$\operatorname{Arg}\operatorname{Det}\widetilde{Y}^{(1)}(t)\Big|_0^\omega = \operatorname{Arg}\operatorname{Det}Y^{(2)}(t)\Big|_0^\omega$$

or, equivalently,

$$\operatorname{Arg}\operatorname{Det}Y^{(1)}(t)\Big|_0^\omega + \operatorname{Arg}\operatorname{Det}W(\tau)\Big|_0^1 = \operatorname{Arg}\operatorname{Det}Y^{(2)}(t)\Big|_0^\omega. \tag{4}$$

Let us turn to the second term on the left-hand side of (4); it is clear that the curve $W(\tau)$ is determined up to a closed curve lying in Ω_P.

Let P be some permutation of p symbols $\oplus$ and q symbols $\ominus$ on the unit circle. Suppose the permutation P can be divided into d groups of $m = n/d$ elements, each of which represents the same permutation of the symbols $\oplus$ and $\ominus$. If, moreover, the permutation does not split into smaller groups, then the number m will be called the **length of the period of the permutation** P (in Fig. 2, $n = 6$, $p = 4$, $q = 2$, $d = 2$, $m = 3$). It is clear that here d is a common divisor of p and n. In particular, if p and n are relatively prime, $d = 1$, and the period length is $m = n$.

Fig. 2

Lemma 1. Let m be the length of the period of the permutation P, and let $W_0(\tau)$, $0 \leqslant \tau \leqslant 1$, be a closed curve lying in the corresponding component Ω_P of J-unitary matrices of stable type. Then

$$(2\pi)^{-1}\operatorname{Arg}\operatorname{Det}W_0(\tau)\Big|_0^1 = km, \tag{5}$$

where k is an integer; moreover, curves exist with any integer k.

---

* As proved in [6],

$$N_{p,q} = \frac{1}{n}\sum_{s=1}^{k}\varphi(d_s)C_{m_s}^{p_s},$$

where $d_1 = 1 < d_2 < ... < d_k$ are the common divisors of the numbers n and p; $p_s = p/d_s$, $m_s = n/d_s$, and $\varphi(d)$ is Euler' s function.

We shall not give the proof of this fact.

Let P be some permutation of p symbols $\oplus$ and q symbols $\ominus$ on a circle with period of length m, and let Ω_P be the corresponding component of J-unitary matrices of stable type. It is easy to construct m nondegenerate curves

$$Y_s(t), \qquad 0 \le t \le \omega, \qquad s = 0, 1, 2, \dots, m-1,$$

with common initial point $Y_s(0) = E_n$ and common endpoint $Y_s(\omega) = Y^0 \in \Omega_P$, for which

$$(2\pi)^{-1} \operatorname{Arg} \operatorname{Det} Y_s(t)\Big|_0^\omega = s.$$

Taking into account (4) and Lemma 1, we can now establish the following assertion.

Theorem 1. *The set $\mathfrak{M}$ of all strongly stable systems of the form (1) decomposes into a finite number of connected components. Each component is characterized by one of the permutations P of the p multipliers of the 1st kind and q multipliers of the 2nd kind on the circle and, in addition, by one integer $0 \le \varkappa \le m-1$, where $m(P)$ is the length of the period of the permutation P.*

In conclusion, let us find the number of all components of the set (the number of stability regions).

Let us note that the total number of permutations of p symbols $\oplus$ and q symbols $\ominus$ on a segment is equal to $C_n^p = n!/p!q!$. Denote by T_{m_s} the number of those permutations on the segment which have period m_s. It is easy to see that exactly $\frac{1}{m_s} T_{m_s}$ permutations on the circle correspond to them. But to each of the indicated permutations, according to Theorem 1, there correspond m_s stability regions. Thus the number of stability regions for which the permutation P has period of length m_s is equal to T_{m_s}, and consequently the total number of stability regions is equal to

$$T_{m_1} + T_{m_2} + \dots + T_{m_k}, \tag{6}$$

where $m_1 > m_2 > \dots > m_k$ ($m_1 = n$) are all possible periods of the permutations. Obviously, the sum (6) is equal to C_n^p (the number of all permutations on the line). We arrive at the following conclusion.

Theorem 2. *The set of all strongly stable systems of the form (1) decomposes into C_n^p stability regions.*

Moscow Institute of Physics and Technology

Received
17 I 1966

CITED LITERATURE

1. V. A. Yakubovich, DAN, **139**, No. 1 (1961).

2. W. A. Coppel, A. Howe, J. Austral. Math. Soc., **5**, No. 2, 169 (1965).

3. V. B. Lidskii, P. A. Frolov, DAN, **161**, No. 4 (1965).

4. M. G. Krein, In memory of A. A. Andronov, Publishing House of the Academy of Sciences of the USSR, 1955, p. 413.

5. I. M. Gelfand, V. B. Lidskii, UMN, **10**, No. 1 (63) (1955).

6. E. Jablonski, J. Math. Pures and Appl. (4), **8**, 331 (1892).

Note: Figure translations are in progress. See original paper for figures.