ON INVERSE THEOREMS OF UNIFORM APPROXIMATION

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Abstract

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ON INVERSE THEOREMS OF UNIFORM APPROXIMATION

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This note is devoted to a certain generalization of the inverse theorems of V. K. Dzyadyk ([1,2]).

Let $\overline{B}$ be a bounded set of points z of type $\mathfrak{M}_s^*$ (see ([3]), pp. 79–81) and let L be the boundary of $\overline{B}$. The complement of L in the extended plane consists of domains $B^j,\ j = 1, 2, \dots, s$, not belonging to $\overline{B}$, and of no more than a countable number of domains $B^j,\ j = s+1, s+2, \dots$, belonging to $\overline{B}$. Further we assume that every point z of the boundary L^j of the domain B^j is accessible from $B^j,\ j = 1, 2, \dots$.

Let a_j be a fixed point in $B^j,\ j = 1, 2, \dots, s$. We shall consider approximation of a function $f(z)$, given on L, by rational functions (r.f.) of the form

$$R(z) = \sum_{j=1}^{s} \sum_{\nu=1}^{n_j} \frac{A_{j,\nu}}{(z - a_j)^\nu} + A, \qquad A_{j,n_j} \neq 0 \tag{1}$$

(if some $a_j = \infty$, then $1/(z - a_j)^\nu$ in (1) must be replaced by z^ν). The number $n = \max_{j=1,2,\dots,s} n_j$ will be called the degree of the r.f. (1).

Let: $g_j(z, \zeta)$ be the Green function of the domain $B^j,\ j = 1, 2, \dots, s$;

$L_\rho^j = \{z :\ g_j(z, a_j) = \ln \rho\},\ \rho > 1;\quad L_\rho = \bigcup_{j=1}^{s} L_\rho^j;\quad d(z, u)$ be the distance from the point $z \in L$ to $L_{1+u},\ u > 0$. Let $A(z),\ z \in L$, be a continuous positive function. Introduce the function $A^*(z)$, putting $A^*(z) = A(z)$ for $z \in L$ and $A^*(z) = \exp G(z, B^j)$ for $z \in B^j,\ j = 1, 2, \dots$, where $G(z, B^j)$ is a function harmonic in B^j and continuous on $\overline{B}^j$ such that $G(z, B^j) = \ln A(z)$ for $z \in L^j$. The function $A^*(z)$ is continuous on the extended plane Z, and

$$\sup_{z \in Z} A^*(z) = \sup_{z \in L} A(z).$$

Put

$$A^*(z,y) = \sup_{|\zeta - z| = y} A^*(\zeta), \qquad z \in L, \quad y > 0.$$

Lemma. Let the r.f. (1) of degree not exceeding n satisfy the condition $|R(z)| \le A(z)$ for $z \in L$. Then:

1) $|R(z)| \le A^*(z)$ for $z \in \overline{B}$ and $|R(z)| \le A^*(z)\rho^n$ for $z \in L_\rho,\ \rho > 1$;

2) for any $u > 0$ and $z \in L$ we have

$$|R^{(\nu)}(z)| \le \nu! e^u A^*(z, d(z, u/n))\, d(z, u/n)^{-\nu}, \qquad \nu = 1, 2, \dots;$$

3) for any $u > 0,\ z_1 \in L,\ z_2 \in L,\ |z_1 - z_2| < d(z_1, u/n)$, we have

$$|R(z_1) - R(z_2)| \le \frac{2}{\pi} K(k) e^u A^*\Big(z_1, d\Big(z_1, \frac{u}{n}\Big)\Big)\, k, \qquad k = \frac{|z_1 - z_2|}{d(z_1, u/n)},$$

where $K(k)$ is the complete elliptic integral of the first kind.

For the ideas of the proof see ([1]), pp. 730-732 (additionally see ([3]), pp. 83, 97).

Let $\omega(x),\ x > 0,\ \omega(0) = 0$, be a nondecreasing continuous function. Put $A(z) = A_u(z) = \omega(d(z,u)),\ z \in L,\ u > 0$. If there exists a number $M \le 1$ such that $A^*(z, d(z,u)) = A_u^*(z, d(z,u)) \le M A_u(z)$ for every $z \in L$ and every $u > 0$, then we write $\omega(x) \in \{\bar{B}, M\}$. If $\omega_1(x) \in$

$\in \{\bar{B}, M_1\}$; $\omega_2(x) \in \{\bar{B}, M_2\}$ and $\alpha > 0$ and $C > 0$ are constants, then $C\omega_1(x)^\alpha \in \{\bar{B}, M_1^\alpha\}$ and $\omega_1(x)\omega_2(x) \in \{\bar{B}, M_1 M_2\}$.

Theorem. Let $f(z)$ be a function defined on the boundary L of the set $\bar{B}$; let $\lambda,\ \lambda > 0$, be a fixed number, r a nonnegative integer; $x^r\omega(x) \in \{\bar{B}, M\}$ and $\omega(\lambda x) \le \lambda' \omega(x)$ for all $x > 0$ and $\lambda' \ge 1$; suppose there exists a sequence of rational functions $R_n(z)$ of degree not exceeding n, $n = m, m+1, \dots$, of the form (1), such that

$$|f(z) - R_n(z)| \le d(z, \lambda/n)^r \omega(d(z, \lambda/n)), \qquad z \in L. \tag{2}$$

Then:

1) if $r = 0$ and $0 < x \le e^{-1} \inf_{z \in L} d(z, \lambda/m) = x_0$, we have

$$\omega(x, f) \le \omega(x, R_m) + C_0 M x \int_x^{l_m} \frac{\omega(t)}{t^2}\, dt, \tag{3}$$

where $l_m = \sup_{z \in L} d(z, \lambda/m)$ and C_0 is an absolute constant ($0 < C_0 < 4.2e^{\lambda_0}$, $\lambda_0 = \max\{(m+1)\lambda/m, \lambda + 2\}$) ($\omega(x, \varphi)$ is the modulus of continuity of the function $\varphi(z)$ on L);

2) if $r > 0$; $\nu = 1, 2, \dots, r$,

$$|f^{(\nu)}(z) - R_n^{(\nu)}(z)| \le C_r^{(\nu)} M \Omega_{r-\nu}(d(z, \lambda/n)), \qquad z \in L, \quad n = m, m+1, \dots, \tag{4}$$

$$\Omega_p(x) = \int_0^x t^{p-1}\omega(t)\,dt, \qquad 0 < C_r^{(\nu)} < \frac{4\nu!\,e^{\lambda_0+q}}{q(e^q-1)}, \qquad q = r - \nu + 1.$$

Remark 1. For $\nu = r$, from (4) we have

$$|f^{(r)}(z) - R_n^{(r)}(z)| \le C_r^{(r)} M \Omega_0\left(d\left(z, \frac{\lambda_*}{n+m}\right)\right), \qquad z \in L, \quad n = m, m+1, \dots,$$

where $\lambda_* = \frac{m+r}{m}\lambda$. If $\Omega_0(x) \in \{\bar{B}, M_*\}$, then the first assertion of the theorem can be applied to $f^{(r)}(z)$.

Remark 2. In the first part of the theorem one can also give an estimate of $\omega(x, f)$ for $x > x_0$.

Proof (see additionally ([2]), pp. 371-374).

1) $r = 0$. Let $0 < x \le x_0$, $z_1 \in L$, $z_2 \in L$, and $|z_1 - z_2| = x$. We have

$$\begin{aligned} f(z_2) - f(z_1) = {} & [f(z_2) - R_{n_1}(z_2)] - [f(z_1) - R_{n_1}(z_1)] \\ & + \sum_{j=1}^{k-1}\{[R_{n_j}(z_2) - R_{n_{j+1}}(z_2)] - [R_{n_j}(z_1) - R_{n_{j+1}}(z_1)]\} \\ & + [R_{n_k}(z_2) - R_{n_k}(z_1)], \end{aligned} \tag{5}$$

where n_j, $m = n_k < n_{k-1} < \dots < n_1 < \infty$, are natural numbers.

We turn to the choice of n_j. Let n be the largest natural number such that

$$\max\{d(z_1, \lambda/n), d(z_2, \lambda/n)\} \ge ex,$$

and suppose, for example, that $d(z_1, \lambda/n) \ge d(z_2, \lambda/n)$. If $d(z_1, \lambda/n) \le e^2 x$, then put $n_1 = n$. If $d(z_1, \lambda/n) > e^2 x$, then put $n_1 = n + 1$ and $n_2 = n$. If some $n_j > m$, $j \ge 1$, has been found, then we find the largest natural n such that $d(z_1, \lambda/n) \ge e d(z_1, \lambda/n_j)$, and if either $d(z_1, \lambda/n) \le e^2 d(z_1, \lambda/n_j)$, or

$$n \ge \frac{m}{m+1} n_j,$$

then put $n_{j+1} = n$. In the contrary case put $n_{j+1} = n + 1$ and $n_{j+2} = n$. We continue this process until we obtain (or are forced to take) some $n_k = m$. For brevity denote

$$d_0 = x, \qquad d_j = d(z_1, \lambda/n_j).$$

There are three possible cases: a) $ed_j \leq d_{j+1} \leq e^2 d_j$; b) $e^2 d_j < d_{j+1}$, but $n_j/n_{j+1} \leq (m+1)/m$; c) $d_{j+1} < ed_j$, but $d_{j+2} > e^2 d_j$ and $n_{j+1}/n_{j+2} \leq (m+1)/m$, or $j + 1 = k$.

Let us estimate the right-hand side in (5). For $\nu = 1, 2$ we have $|f(z_\nu) - R_{n_1}(z_\nu)| \leqslant \omega(d(z_\nu, \lambda/n_1)) \leqslant \omega(d_1)$. If $d_1 < ex$, then $\omega(d_1) \leqslant \omega(ex)$, and if $ex \leqslant d_1 \leqslant e^2 x$, then $\omega(d_1) \leqslant \dfrac{d_1}{ex}\omega(ex) \leqslant e\omega(ex)$, and, consequently,

$$\big|[f(z_2) - R_{n_1}(z_2)] - [f(z_1) - R_{n_1}(z_1)]\big| \leqslant 2e\omega(ex).$$

$$|R_{n_j}(z_1) - R_{n_{j+1}}(z_1)| \leqslant |f(z_1) - R_{n_j}(z_1)| + |f(z_1) - R_{n_{j+1}}(z_1)| \leqslant 2\omega(d_{j+1}),$$

and, by virtue of the lemma (in our case $k \leqslant e^{-1}$ and $\dfrac{2}{\pi}K(k) < 1.05$), we have

$$I_j = \big|[R_{n_j}(z_2) - R_{n_{j+1}}(z_2)] - [R_{n_j}(z_1) - R_{n_{j+1}}(z_1)]\big| \leqslant$$

$$\leqslant 2.1\, M e^{\lambda u} \omega(d_{j+1}) d(z_1, \lambda u/n_j)^{-1} x$$

for $0 < u \leqslant n_j/n_{j+1}$. If $n_j/n_{j+1} \leqslant (m+1)/m$, then put $u = n_j/n_{j+1}$. In the contrary case put $u = 1$ and use the fact that $d_{j+1} \leqslant e^2 d_j$. In both cases we obtain the estimate $I_j \leqslant 2.1 e^{\lambda_0} M \omega(d_{j+1}) d_{j+1}^{-1} x$.

From what has been stated and from (5) we have

$$|f(z_2) - f(z_1)| \leqslant 2e^2 \frac{\omega(ex)}{ex} x + 2.1 e^{\lambda_0} M x \sum_{j=2}^{k} \frac{\omega(d_j)}{d_j} + \omega(x, R_m). \qquad (6)$$

Let us estimate $\omega(d_j)/d_j$. If $d_j \geqslant ed_{j-1}$, then

$$\int_{d_{j-1}}^{d_j} \frac{\omega(t)}{t^2}\, dt \geqslant \frac{\omega(d_j)}{d_j} \int_{d_{j-1}}^{d_j} \frac{dt}{t} \geqslant \frac{\omega(d_j)}{d_j}.$$

If $d_j < ed_{j-1}$, then $d_{j-1} \geqslant ed_{j-2}$, and therefore

$$\int_{d_{j-2}}^{d_j} \frac{\omega(t)}{t^2}\,dt \geqslant \frac{\omega(d_j)}{d_j} \int_{d_{j-2}}^{d_{j-1}} \frac{dt}{t} \geqslant \frac{\omega(d_j)}{d_j}.$$

$$\int_{x}^{ex} \frac{\omega(t)}{t^2}\,dt \geqslant \frac{\omega(ex)}{ex}.$$

Now from (6) it follows that

$$|f(z_2) - f(z_1)| \leqslant \omega(x, R_m) + 4.2e^{\lambda_0} Mx \int_{x}^{d_k} \frac{\omega(t)}{t^2}\,dt,$$

and, since $d_k \leqslant l_m$, the case $r = 0$ has been considered.

2) $r > 0$. Choose a natural number $n_0 \geqslant m$ and represent $f^{(\nu)}(z)$ as

$$f^{(\nu)}(z) - R_{n_0}^{(\nu)}(z) = \sum_{j=1}^{\infty} [R_{n_j}^{(\nu)}(z) - R_{n_{j-1}}^{(\nu)}(z)], \quad z \in L, \tag{7}$$

where $m \leqslant n_0 < n_1 < n_2 < \cdots$ are natural numbers chosen analogously to what was set out in the first part of the proof. Further, relying on the lemma, we estimate the right-hand side in (7) and obtain (4).

For the functions $\omega(x)$ and sets $\overline{B}$ considered by V. K. Dzyadyk, it was in fact proved by him that $\omega(x) \in \{\overline{B}, M\}$. It is not difficult to indicate broader classes of sets $\overline{B}$ for which $\omega(x) \in \{\overline{B}, M\}$ ($\omega(0) = 0$, $\omega(\lambda x) \leqslant \lambda\omega(x)$ for all $x > 0$ and $\lambda \geqslant 1$), but a complete analysis could not be carried out.

The foregoing can easily be somewhat generalized.

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CITED LITERATURE

1. V. K. Dzyadyk, *Izv. AN SSSR, Ser. Mat.*, **23**, 697 (1959).
2. V. K. Dzyadyk, *Ukr. Mat. Zh.*, **15**, No. 4 (1963).
3. V. I. Smirnov, N. A. Lebedev, *Constructive Theory of Functions of a Complex Variable*, "Nauka," 1964.

Note: Figure translations are in progress. See original paper for figures.