



---

Soviet-era science, translated into English

# SOME ESTIMATES FOR A SINGULAR INTEGRAL \

MATHEMATICS

1966

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.41960>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.554

*MATHEMATICS*

**A. A. BABAEV**

## **SOME ESTIMATES FOR A SINGULAR INTEGRAL \***

*(Presented by Academician I. N. Vekua, January 14, 1966)*

Let the function  $u(x)$  be defined and continuous on the interval  $(a, b)$ . Introduce the functions

$$\Omega(u, r_1, r_2) = \max_{x \in [a+r_1, b-r_2]} |u(x)|,$$

$$\omega(u, \tau, r_1, r_2) = \max_{\substack{|x_1 - x_2| \leq \tau \\ x_1, x_2 \in [a+r_1, b-r_2]}} |u(x_1) - u(x_2)|,$$

where  $0 < r_1, r_2, r_1 + r_2 < b - a$ ,  $0 < \tau \leq b - a - r_1 - r_2$ . Denote

$$v(x) = \int_a^b \frac{u(s)}{s - x} ds,$$

where the integral is understood in the sense of the Cauchy principal value.

**Theorem 1.** If the integrals

$$\begin{aligned} & \int_0^{\frac{b-a}{2}} \Omega(u, t, (b-a)/2) dt, & \int_0^{\frac{b-a}{2}} \Omega(u, (b-a)/2, t) dt, \\ & \int_0^{\frac{b-a}{4}} \frac{\omega(u, t, r_1/2, (b-a)/4)}{t} dt, & \int_0^{\frac{b-a}{4}} \frac{\omega(u, t, (b-a)/4, r_2/2)}{t} dt \end{aligned}$$

converge, then the estimate

$$\Omega(v, r_1, r_2) \leq C_1 \left[ \frac{1}{r_1} \int_0^{r_1/2} \Omega(u, t, (b-a)/2) dt + \frac{1}{r_2} \int_0^{r_2/2} \Omega(u, (b-a)/2, t) dt \right. \\ \left. + \int_0^{r_1/2} \frac{\omega(u, t, r_1/2, (b-a)/4)}{t} dt + \int_0^{r_2/2} \frac{\omega(u, t, (b-a)/4, r_2/2)}{t} dt \right. \\ \left. + \int_{r_1/2}^{(b-a)/2} \frac{\Omega(u, t(b-a)/4)}{t} dt + \int_{r_2/2}^{(b-a)/2} \frac{\Omega(u, (b-a)/4, t)}{t} dt \right],$$

where  $C_1$  is a constant independent of  $u(x)$ .

**Theorem 2.** If the integrals

$$\int_0^{\delta} \omega(u, r_2/2, r_1/2, t) dt, \quad \int_0^{\delta} \omega(u, r_1/2, t, r_2/2) dt, \quad \int_0^{\delta} \frac{\omega(u, t, r_1/2, r_2/2)}{t} dt,$$

converge, then the estimate

$$\omega(v, \delta, r_1, r_2) \leq C_2 \left[ \delta \int_{\varepsilon}^{(b-a)/2} \frac{\omega(u, t, r_1/2, r_2/2)}{t^2} dt \right. \\ \left. + \frac{\delta}{r_1^2} \int_0^{r_1/2} \omega(u, r_1/2, t, r_2/2) dt + \frac{\delta}{r_2^2} \int_0^{r_2/2} \omega(u, r_2/2, r_1/2, t) dt + \right.$$

\* Presented at the All-Union interuniversity conference on the application of methods of functional analysis to the solution of nonlinear problems, Baku, November 1965.

$$\left. + \int_0^{\delta} \frac{\omega(u, t, r_1/2, r_2/2)}{t} dt + \left( \frac{\delta}{r_1} + \frac{\delta}{r_2} \right) \Omega(u, r_1, r_2) \right],$$

where  $\varepsilon = \min\{\delta, r_1/2, r_2/2\}$ , and  $C_2$  is a constant independent of  $u(x)$ .

Let us consider some constructions based on the preceding estimates.

Suppose: a)  $\varphi(\delta)$  is continuous, positive, and monotonically increasing for  $0 < \delta \leq b-a$ ,  $\lim_{\delta \rightarrow 0} \varphi(\delta) = 0$ , and  $\lim_{\delta \rightarrow 0} \varphi(\delta)/\delta > 0$ ; b) the positive continuous functions  $\varphi_1(r_1)$ ,  $\tilde{\varphi}_1(r_1)$ ,  $\varphi_2(r_2)$ ,  $\tilde{\varphi}_2(r_2)$ ,  $0 < r_1, r_2 < b-a$ , increase monotonically and have limit zero at zero.

**Definition.** We shall say that  $u(x)$ , defined in  $(a, b)$ , belongs to the class  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$  if there exist constants  $A_1, A_2 > 0$  such that, for all possible  $r_1, r_2, \delta$ ,

$$\Omega(u, r_1, r_2) \leq A_1 \frac{\tilde{\varphi}_1(r_1)}{r_1} \frac{\tilde{\varphi}_2(r_2)}{r_2}, \quad \omega(u, \delta, r_1, r_2) \leq A_2 \varphi(\delta) \frac{\varphi_1(r_1)}{r_1} \frac{\varphi_2(r_2)}{r_2}.$$

It is obvious that  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$  is a linear system over the field of real numbers and, moreover, it is infinite-dimensional, since, by virtue of the condition  $\lim_{\delta \rightarrow 0} \varphi(\delta)/\delta > 0$ , it contains the class of functions satisfying the Lipschitz condition on  $[a, b]$ .

With the introduction in  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$  of the norm

$$\|u\|_{\varphi, \psi_1, \psi_2}^{\tilde{\psi}_1, \tilde{\psi}_2} = \sup_{r_1, r_2} \frac{\Omega(u, r_1, r_2) r_1 r_2}{\tilde{\varphi}_1(r_1) \tilde{\varphi}_2(r_2)} + \sup_{r_1, r_2, \delta} \frac{\omega(u, \delta, r_1, r_2) r_1 r_2}{\varphi(\delta) \varphi_1(r_1) \varphi_2(r_2)}$$

it becomes a Banach space.

The following theorem partially solves the problem of classifying  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ .

**Theorem 3.** Let  $\varphi(\delta)$ ,  $\psi(\delta)$  satisfy condition a), and let  $\varphi_1(r_1)$ ,  $\tilde{\varphi}_1(r_1)$ ,  $\varphi_2(r_2)$ ,  $\tilde{\varphi}_2(r_2)$  and  $\psi_1(r_1)$ ,  $\tilde{\psi}_1(r_1)$ ,  $\psi_2(r_2)$ ,  $\tilde{\psi}_2(r_2)$  satisfy condition b).

1) If

$$\varphi(\delta) \sim \psi(\delta)^*, \quad \varphi_1(r_1) \sim \psi_1(r_1), \quad \tilde{\varphi}_1(r_1) \asymp \tilde{\psi}_1(r_1),$$

$$\varphi_2(r_2) \sim \psi_2(r_2), \quad \tilde{\varphi}_2(r_2) \sim \tilde{\psi}_2(r_2),$$

then  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$  and  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\psi}_1, \tilde{\psi}_2}$  coincide.

2) If

$$\lim_{\delta \rightarrow 0} \frac{\varphi(\delta)}{\psi(\delta)} = 0, \quad \lim_{r_1 \rightarrow 0} \frac{\varphi_1(r_1)}{\psi_1(r_1)} = 0, \quad \lim_{r_2 \rightarrow 0} \frac{\varphi_2(r_2)}{\psi_2(r_2)} = 0,$$

$$\lim_{r_1 \rightarrow 0} \frac{\tilde{\varphi}_1(r_1)}{\tilde{\psi}_1(r_1)} = 0, \quad \lim_{r_2 \rightarrow 0} \frac{\tilde{\varphi}_2(r_2)}{\tilde{\psi}_2(r_2)} = 0,$$

then  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$  is a proper part of  $H_{\psi, \psi_1, \psi_2}^{\tilde{\psi}_1, \tilde{\psi}_2}$ .

For the subsequent arguments the following will be useful.

**Definition.** Let a positive, continuous function  $f(t)$  increase monotonically on  $(0, a]$  and let  $\lim_{t \rightarrow 0} f(t) = 0$ . If, for some  $c_1 > 1$ ,

$$\lim_{t \rightarrow 0} f(c_1 t)/f(t) > 1,$$

then one says that the function  $f(t)$  satisfies condition (L), and if, for some  $c_2 > 1$ ,

$$\lim_{t \rightarrow 0} f(c_2 t)/f(t) < c_2,$$

then one says that  $f(t)$  satisfies condition (L<sub>1</sub>).

---

\* Positive functions  $f(x)$  and  $g(x)$ , defined on  $X$ , are called equivalent ( $f(x) \sim g(x)$ ) if, for some  $B_1, B_2 > 0$  and every  $x \in X$ ,  $B_1 f(x) \leq g(x) \leq B_2 f(x)$ .

With the help of Theorems 1 and 2 one proves

**Theorem 4.** *Let the functions  $\varphi(\delta)$ ,  $\tilde{\varphi}_1(r_1)$ ,  $\tilde{\varphi}_2(r_2)$  satisfy conditions (L) and (L<sub>1</sub>), and let the functions  $\varphi_1(r_1)$ ,  $\varphi_2(r_2)$  satisfy condition (L). Suppose, furthermore, that*

$$\varphi(t)\varphi_1(t) \sim \tilde{\varphi}_1(t), \quad \varphi(t)\varphi_2(t) \sim \tilde{\varphi}_2(t).$$

Then the operator

$$Au = \int_a^b \frac{u(s)}{s-x} ds$$

acts in  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$  and is bounded.

The class  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ , corresponding to the functions  $\varphi(\delta) = \delta^\gamma$ ,  $\tilde{\varphi}_1(r_1) = r_1^{1-\alpha}$ ,  $\tilde{\varphi}_2(r_2) = r_2^{1-\beta}$ ,  $\varphi_1(r_1) = r_1^{1-\alpha-\gamma}$ ,  $\varphi_2(r_2) = r_2^{1-\beta-\gamma}$  ( $0 < \alpha, \beta, \gamma$ ,  $\alpha + \gamma$ ,  $\beta + \gamma < 1$ ), will be denoted by  $H_{\alpha, \beta, \gamma}$ . The invariance of  $H_{\alpha, \beta, \gamma}$  with respect to the operator  $A$  was proved by another method in (2). Arguing exactly as in (3), with the help of Theorems 3 and 4, one can show that there exists an infinite set of various classes  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ , different from the classes  $H_{\alpha, \beta, \gamma}$  and invariant with respect to the operator  $A$ .

With the help of Theorem 4 one can study the question of smoothness of solutions of linear and nonlinear singular integral equations. We give one theorem of this type for nonlinear singular integral equations, which is proved by the method set forth in papers (4, 5).

**Theorem 5.** *Let the functions  $\varphi(\delta)$ ,  $\varphi_1(r_1)$ ,  $\varphi_2(r_2)$ ,  $\tilde{\varphi}_1(r_1)$ ,  $\tilde{\varphi}_2(r_2)$  satisfy the conditions of Theorem 4, and let the function  $f(s, u)$ , for arbitrary  $s_1, s_2 \in$*

$[a + r_1, b - r_2]$ ,  $0 < r_1, r_2$ ,  $r_1 + r_2 < b - a$  and  $u_1, u_2 \in (-\infty, +\infty)$ , satisfy the condition

$$|f(s_1, u_1) - f(s_2, u_2)| \leq D \left[ \varphi(|s_1 - s_2|) \frac{\varphi_1(r_1)}{r_1} \frac{\varphi_2(r_2)}{r_2} + |u_1 - u_2| \right],$$

where  $D$  is a constant. Suppose, furthermore, that  $f(s, 0) \in H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ .

Then, for small  $\lambda$ , the equation

$$u(x) = \lambda \int_a^b \frac{f[s, u(s)]}{s - x} ds$$

has a unique solution in  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ , and this solution can be found by the method of successive approximations, starting from any element of  $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ . The successive approximations converge uniformly on every closed subinterval interior to  $(a, b)$ .

Azerbaijan State University  
named after S. M. Kirov

Received  
24 XII 1965

## REFERENCES

1. N. K. Bari, S. B. Stechkin, *Tr. Moskov. matem. obshch.*, **5**, 483 (1956).
2. A. I. Guseinov, *Izv. AN SSSR, ser. matem.*, **12**, No. 2 (1948).
3. A. A. Babaev, *Uch. zap. Azerb. gos. univ., ser. fiz.-matem. nauk*, No. 4 (1963).
4. A. A. Babaev, *Uch. zap. Azerb. gos. univ., ser. fiz.-matem. i khim. nauk*, No. 2 (1960).
5. A. A. Babaev, *Uch. zap. Azerb. gos. univ., ser. fiz.-matem. i khim. nauk*, No. 1 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*