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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON SOME FUNCTION-THEORETIC SIEVE METHODS IN NUMBER THEORY\***

*(Presented by Academician Yu. V. Linnik, 3 X 1966)*

The Hardy–Littlewood–Vinogradov method is still the strongest method of additive number theory. This method consists in the following: we seek the generating function in the form of a power series, represent the desired quantity as a curvilinear integral of this function and, decomposing the path of integration into small arcs, replace on each of these arcs the generating function by suitable “elementary” functions. In doing so, in the case of non-extendability of the generating function, the contour of integration is chosen “not far” from the boundary of the domain of existence of this function. The elementary functions can be chosen in various ways. In problems with prime summands, the construction of elementary functions leads to the study of properties of Dirichlet functions  $L(s, k, \chi)$  (I use this notation, since in the case of variable  $k$  the usual notation  $L(s, \chi)$  is ambiguous; similarly I denote the characters by  $\chi(\mu, k)$  instead of the usual  $\chi(\mu)$ ). As I found in <sup>(1)</sup>, in the twin-prime problem (and in some other problems of an “indefinite” type) one can avoid the indirect route described above by returning to generating functions in the form of Dirichlet series. Indeed, a simple application of the sieve method to an infinite Dirichlet series readily leads, as was shown in my paper <sup>(1)</sup>, to the formula

$$\sum_{\substack{\mu \leq N-2 \\ \mu \text{ odd}}} \frac{\Lambda(\mu)\Lambda(\mu+2)}{\mu^s} + \sum_{4 \leq 2^\lambda \leq N-2} \frac{\log 2 \log(2^\lambda - 1 + 1)}{2^{\lambda s}} + \sum_{\mu > N-2} \frac{l_\mu}{\mu^s} = \quad (1)$$

$$= \sum_{\substack{k < N \\ k \text{ odd}}} \frac{\mu(k) \log k}{\varphi(k)} \sum_{\chi \bmod k} \bar{\chi}(-2, k) \frac{L'}{L}(s, k, \chi),$$

where  $\sigma > 1$ ,  $l_\mu$  are certain numbers;  $N \geq 10$  is an arbitrary integer. The defect of every sieve method is the appearance of “unpleasant” summands; in our case these summands are concentrated in the sum

$$\sum_{\mu > N-2} \frac{l_\mu}{\mu^s}.$$

The novelty and usefulness of this function-theoretic sieve is that, since (1) holds in a certain half-plane, any formula for the coefficients of a general Dirichlet series with  $\mu \leq N-2$  makes it possible to eliminate all the  $l_\mu$  at once; this makes it possible to apply the technique of contour integration successfully, since the integrals over vertical straight lines and all functions on the right-hand side of (1) are meromorphic in the whole plane.

Leaving aside a general investigation of this method, I shall return to the consideration of additive problems of a “definite” type, examples of which are the binary Goldbach problem or the Hardy–Littlewood problem, recently solved by Yu. V. Linnik, on the representability of integers  $N$

\* Dedicated to the sixtieth birthday of Professor A. O. Gelfond.

in the form  $p+x^2+y^2$  with prime  $p$ . Recently I discovered that in some problems of this type one can also avoid an analogous roundabout route. Here matters are even simpler than before, since we rely on the uniqueness theorem for Dirichlet series. To make this precise, let us consider the representability of an integer  $N$  in the form

$$N = \mu + f(x_1, x_2, \dots, x_k) = \mu + f(x), \quad (2)$$

where  $\mu$  is a prime number;  $x_1, x_2, \dots, x_k$  range over certain sets  $\Pi_1, \Pi_2, \dots, \Pi_k$  of positive integers;  $f$  is a positive-definite polynomial, so that the number of solutions  $h(\nu)$  of the equation  $f(x) = \nu$  ( $x_i \in \Pi_i$ ) is finite for every positive integer  $\nu$  and even  $h(\nu) < \nu^c$ . Suppose, moreover, that  $h(\nu)$  is such that one can investigate in detail the function-theoretic properties of the functions

$$G_N(w, k, \chi) = \sum_{(\nu, N)=1} \frac{h(\nu)\chi(\nu, k)}{\nu^w}. \quad (3)$$

(This condition is satisfied, for example, if  $h(\nu)$  is a multiplicative function, which is of course true for  $f = x_1^2 + x_2^2$ .)

Let, further,  $K(w)$  be a function regular in the half-plane  $\operatorname{Re} w \geq -3$  and, for example, such that the function

$$\frac{1}{2\pi i} \int_{(3)} K(w)x^w dw = g(x) \quad (4)$$

is equal to zero for  $0 \leq x \leq 1$  and is nonnegative for  $x \geq 1$ . As  $K(w)$  one may choose, for instance, the function

$$K(w) = \frac{1 - e^{-aw}}{w(1 + w/b)^\lambda},$$

where  $b > 3$ ,  $a > 0$ ,  $\lambda \geq 3$  is an integer. Put, further,  $s = \sigma + it$ , and suppose that  $\sigma > 2$  and  $\eta > 1$  are chosen so that  $\sigma - \eta > 1$ . Consider the function

$$F_N(s) = \sum_{k \leq N} \frac{\mu(k) \log k}{\varphi(k)} \sum_{\chi \bmod k} \frac{1}{2\pi i} \int_{(\eta)} L(s-w, k, \chi) G_N(w, k, \chi) K(w) dw, \quad (5)$$

where  $N$  is taken from (2). In the integrand one may, evidently, replace the functions  $L(s-w, k, \chi)$  and  $G_N(w, k, \chi)$  by the Dirichlet series representing them and integrate the resulting series term by term, which easily gives a representation of  $F_N(s)$  as a series  $\sum_{\mu} a_{\mu} \mu^{-s}$ , where

$$a_{\mu} = - \sum_{\substack{\nu < \mu \\ (\nu, N)=1}} h(\nu) g\left(\frac{\mu}{\nu}\right) \sum_{\substack{(k, \mu)=(k, \nu)=1 \\ k|\mu-\nu}} \mu(k) \log k, \quad (6)$$

whence it follows at once that the series  $\sum_{\mu} a_{\mu} \mu^{-s}$  converges (and indeed absolutely) in the half-plane  $\sigma > c$ . It is easy to see that

$$a_N = \sum_{\substack{\nu < N \\ (\nu, N)=1}} h(\nu) g\left(\frac{N}{\nu}\right) \Lambda(N - \nu).$$

In most cases  $K(w)$  is such that  $g(N/\nu)$  is “close” to 1, except for a small number of values of  $\nu$ , so that, on the one hand, we have

$$a_N \sim \sum_{\substack{p < N \\ (p, N)=1}} \log p h(N-p) \sim \log N \sum_{p+f(x)=N} 1, \quad (7)$$

while, on the other hand, in (5) one may move the contour of integration to the left. We obtain first of all a sum of residues. In many cases the function

$G_N(w, k, \chi)$  has a simple pole at  $w = 1$ ,  $\chi = \chi_0$  with residue  $B_k(N)$ ; the corresponding term in the sum of residues has the form

$$- \sum_{k < N} \frac{\mu(k) \log k}{\varphi(k)} B_k(N) L(s-1, k, \chi_0). \quad (8)$$

This expression can again be represented in the form of a Dirichlet series  $\sum_{\mu} b_{\mu} \mu^{-s}$ , where

$$b_{\mu} = -\mu \sum_{\substack{k < N \\ (k, \mu)=1}} \frac{\mu(k) \log k}{\varphi(k)} B_k(N), \quad (9)$$

whence it follows again that the series converges (absolutely) in the half-plane  $\sigma > c$ . In the case where, at  $w = 1$ , there is a multiple pole, the necessary changes are obvious; the additional terms arising from residues at  $w = \rho$  (if any) have a form analogous to (8), and are represented by Dirichlet series  $\sum_{\mu} b_{\mu}(\rho)\mu^{-s}$ . A suitable choice of  $K(w)$  guarantees the inequality

$$\sum_{\rho} \sum_{\mu} \frac{|b_{\mu}(\rho)|}{\mu^{\sigma}} < \infty$$

for  $\sigma > 0$ . Let the new path of integration be the line  $\operatorname{Re} w = \theta < \eta$ ; the resulting integral is equal to  $\sum_{\mu} d_{\mu}\mu^{-s}$ , where

$$d_{\mu} = - \sum_{\substack{k < N \\ (k, \mu) = 1}} \frac{\mu(k) \log k}{\varphi(k)} \frac{1}{2\pi i} \int \mu^w G_N(w, k, \bar{\chi}) K(w) dw, \quad (10)$$

so that in most cases the series  $\sum_{\mu} d_{\mu}\mu^{-s}$  converges (absolutely) in some half-plane. By the uniqueness theorem, the coefficients of the Dirichlet series obtained as a result are equal to the corresponding coefficients of the original series. In particular, for  $\mu = N$  we have

$$a_N = b_N + \sum_{(\rho)} b_N(\rho) + d_N. \quad (11)$$

This equality, taking into account relation (7), gives an asymptotic formula for the number of solutions of equation (2).

As a first application of this method I investigated the number  $\nu_2(N)$  of representations of an even number  $N$  in the binary Goldbach problem. Denote the nontrivial zeros of  $L(w, k, \chi)$  by  $\rho = \beta + i\gamma$ ; let  $\varepsilon(x)$  be a function tending to zero arbitrarily slowly as  $x \rightarrow \infty$ , and let  $m = [\varepsilon(N) \log N]$ . I proved the following theorem.

**Theorem.** As  $N \rightarrow \infty$  the formula holds

$$\begin{aligned} \nu_2(N) = & (1 + o(1)) \frac{N}{\log^2 N} 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p>2 \\ p|N}} \frac{p-1}{p-2} \\ & + \frac{(1 + o(1))}{\log^2 N} \sum_{\substack{k < N \\ (k, N) = 1}} \frac{\mu(k) \log k}{\varphi(k)} \sum_{\chi \bmod k} \bar{\chi}(N, k) \sum_{\substack{(\rho(\chi)) \\ |\gamma| \leq \exp(2/\varepsilon(N)) \log N}} \frac{N^{\rho} - N^{\rho/100}}{\rho(1 + \rho/\log N)^{m+1}}. \end{aligned} \quad (12)$$

In the principal term of formula (12) one can recognize the expression heuristically predicted by Hardy and Littlewood in their fundamental paper (2) (see, in

particular, p. 32); the corresponding changes in the argument lead to an analogous formula for the number of twin primes not exceeding  $x$ . These formulas, it seems to me, will play a role in the investigation of the corre-

of the corresponding problems the same role that Riemann's "exact" formula for the number of primes played in the theory of prime numbers. The situation appears even more favorable, since the principal term in (12) (and in analogous formulas in other problems) can probably be investigated, for example, by means of Yu. V. Linnik's "large sieve" method; in any case, this formula indicates what problems arise on the path to a complete solution of the problem. The full proof of the theorem formulated above, as well as other applications of the methods described, will be published in the journal *Acta Arithmetica*.

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- <sup>1</sup> P. Turan, Proc. Lond. Math. Soc., 288 (1965).
- <sup>2</sup> G. H. Hardy, J. E. Littlewood, Acta math., 44, 1 (1922).

*Note: Figure translations are in progress. See original paper for figures.*

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