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Abstract

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MATHEMATICS

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SOME LIMIT THEOREMS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A RANDOM RIGHT-HAND SIDE

(Presented by Academician A. N. Kolmogorov on 14 IX 1965)

In many physical and, in particular, radio-engineering problems it is of interest to study random processes that are solutions of differential equations whose right-hand side contains randomness. In this case it is often possible to single out in the right-hand side of the equation a small parameter ε, which may characterize the smallness of the random perturbation, its "correlation time," etc. (see ([1])). In the present note we study the asymptotic behavior of such random processes as $\varepsilon \to 0$.

1. Let $F(x, t, \omega, \varepsilon)$ be a function with values in l-dimensional Euclidean space E^l, defined for $x \in E^l$, $t \geq 0$, $\omega \in \Omega$, $\varepsilon \geq 0$; here Ω is the space of elementary events, on the σ-algebra $\mathfrak{A}$ of measurable subsets of which a probability measure P is given. Suppose that $F(x, t, \omega, \varepsilon)$, for fixed x, ε, is a random process measurable in t, ω, satisfies the Lipschitz condition

$$|F(x_2, t, \omega, \varepsilon) - F(x_1, t, \omega, \varepsilon)| < L|x_2 - x_1| \tag{1}$$

and, for all $t > 0$, the condition

$$\mathbf{P}\left\{\int_0^t |F(0, s, \omega)|\, ds < \infty\right\} = 1. \tag{2}$$

When these requirements are fulfilled, the problem

$$dx/dt = \varepsilon F(x, t, \omega, \varepsilon); \qquad x(0) = x_0, \tag{3}$$

has a unique solution, a random process $x_\varepsilon(t, \omega)$ that is continuous with probability 1.

The asymptotic behavior, as $\varepsilon \to 0$, of the solution $x_\varepsilon(t, \omega)$ of problem (3) has been considered, for example, in ([1–3]).

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In ([3]) it is proved that the function $x_\varepsilon(t,\omega)$ on a time interval of order $O(1/\varepsilon)$ can be uniformly approximated by the solution of the problem

$$\frac{dx}{dt} = \varepsilon\Phi^{(0)}(x); \qquad x(0) = x_0 \qquad \left(\Phi^{(0)}(x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T MF^{(0)}(x,t,\omega)\,dt\right),$$

if, uniformly in x, t, ω as $\varepsilon \to 0$, the relation $F(x,t,\omega,\varepsilon) = F^{(0)}(x,t,\omega) + o(1)$ holds, and the process $F^{(0)}(x,t,\omega)$ satisfies the law of large numbers.

It follows from this result that, when $\Phi^{(0)}(x) \equiv 0$, the random process $x_\varepsilon(\varepsilon t,\omega)$ converges in probability to zero as $\varepsilon \to 0$. R. L. Stratonovich in ([1]) first drew attention to the fact that in the still "slower" time $\tau = \varepsilon^2 t$ there is convergence to a Markov process. However, some points in the proof of this assertion given in ([2]), when F is a stationary-in-time random process, seem to us unconvincing. We shall consider this problem under other assumptions and by another method.

Let the function $F(x,t,\omega,\varepsilon)$ satisfy the conditions:

B$_1$. Uniformly in x, t, ω, except perhaps for a set of ω-values of probability 0, the relations

$$F(x,t,\omega,\varepsilon) = F^{(0)}(x,t,\omega) + \varepsilon F^{(1)}(x,t,\omega) + o(\varepsilon) \quad (\varepsilon \to 0),$$

$$|F^{(i)}|,\ |\partial F^{(i)}/dx|,\ |\partial^2 F^{(i)}/\partial x_j \partial x_k| < C \quad (i = 0,1;\ j,k = 1,\dots,l).$$

B$_2$. There exists a family of σ-algebras N_s^t $(0 \le s \le t \le \infty)$ of subsets of Ω such that $N_s^t \subset \mathfrak{A}$; $N_s^t \subset N_{s_1}^{t_1}$ if $s_1 \le s,\ t \le t_1$, and for all $x \in E_n,\ t \ge 0$ the random variables $F^{(i)}(x,t,\omega)$ are N_t^t-measurable. (For example, if $F^{(i)}(x,t,\omega) = F^{(i)}(x,\xi(t,\omega))$, then one may take as N_s^t the σ-algebra of events generated by events of the form $\{\xi(u,\omega) \in A\}$, $s \le u \le t$.) Moreover, for any $t \ge 0$, $B \in N_{t+\tau}^\infty$, and for some function $\beta(\tau)$ such that, as $\tau \to \infty$, the function $\tau^6\beta(\tau) \downarrow 0$, with probability 1,

$$|P\{B/N_0^t\} - P(B)| < \beta(\tau). \tag{4}$$

(Condition (4) was considered by Ibragimov in ([4,5]).)

B$_3$. Uniformly in $x, t_0 > 0$ the limits

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} MF^{(1)}(x,t,\omega)\,dt = \Phi^{(1)}(x),$$

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$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \operatorname{cov}(F_j^{(0)}(x,s,\omega), F_k^{(0)}(x,t,\omega))\, ds\, dt = a_{jk}(x), \tag{5}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} ds \int_{t_0-T}^{s} M\left\{\frac{\partial F^{(0)}}{\partial x}(x,s,\omega) F^{(0)}(x,t,\omega)\right\} dt = K(x),$$

exist, and the integrals

$$\int_0^T MF^{(0)}(x,t,\omega)\, dt \quad \text{and} \quad \int_0^T \frac{\partial}{\partial x} MF^{(0)}(x,t,\omega)\, dt$$

are bounded uniformly in x, T.

B$_4$. There exists a sequence $T_n \to \infty$, growing no faster than a geometric progression and such that, as $n \to \infty$,

$$\delta(T_n) = \sup_{x\in E^l,\ t_0>T_n} \left| T_n^6 \int_{t_0}^{t_0+T_n} MF^{(0)}(x,t,\omega)\, dt \right| \downarrow 0.$$

Theorem 1. *If the function $F(x,t,\omega,\varepsilon)$ satisfies conditions B_1–B_2, then the process $x_\varepsilon(\varepsilon^2 t)$ on the time interval $0 \le \varepsilon^2 t \le \tau_0$ converges weakly as $\varepsilon \to 0$ to a continuous, with probability 1, Markov process $X^{(0)}(t,\omega)$, for which*

$$M\{\Delta X^{(0)}(t,\omega) \mid X^{(0)}(t,\omega) = x\} = (K(x) + \Phi^{(1)}(x))\Delta t + o(\Delta t),$$

$$M\{\Delta X_j^{(0)}(t,\omega)\Delta X_k^{(0)}(t,\omega) \mid X^{(0)}(t,\omega) = x\} = a_{jk}(x)\Delta t + o(\Delta t).$$

From Theorem 1 one can easily derive

Theorem 2. *Let the function F satisfy conditions B_1, B_2, and let*

$$F^{(i)}(x,t,\omega) = F^{(i)}(x,\xi(t,\omega)),$$

where $\xi(t,\omega)$ is a random process periodic with period θ. Suppose also that the condition

$$\int_0^\theta MF^{(0)}(x,t,\omega)\, dt = 0 \tag{6}$$

is satisfied.

Then the conclusion of Theorem 1 is valid, where $a_{jk}(x)$, $\Phi^{(1)}(x)$, and $K(x)$ are computed by the formulas

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$$\Phi^{(1)}(x) = \frac{1}{\theta}\int_0^\theta MF^{(1)}(x,t,\omega)\,dt;$$

$$a_{jk}(x) = \frac{1}{\theta}\int_0^\theta ds\int_{-\infty}^{+\infty}\operatorname{cov}\left(F_j^{(0)}(x,s,\omega), F_k^{(0)}(x,t,\omega)\right)dt,$$

$$K(x) = \frac{1}{\theta}\int_0^\theta ds\left[\int_{-\infty}^{\theta}\operatorname{cov}\left(\frac{\partial F^{(0)}}{\partial x}(x,s,\omega), F^{(0)}(x,s+u,\omega)\right)du + \int_0^s \frac{\partial MF^{(0)}}{\partial x}(x,s)MF^{(0)}(x,t)\,dt\right]. \tag{7}$$

Remark 1. For a stationary process F, formula (7) is somewhat simplified (since condition (6) becomes the condition $MF^{(0)}(x,t,\omega) \equiv 0$, and in (7) one can pass to the limit as $\theta \to 0$). For this case they were obtained by Stratonovich in ([1],[2]).

Remark 2. Relying on Theorem 1, one can show that the conclusion of Theorem 2 is also valid in the case when the process $\xi(t,\omega)$ is not periodic, but only converges to a periodic one in a sufficiently weak sense as $t \to \infty$.

Remark 3. Theorem 1 can be regarded as a generalization of the central limit theorem for random processes satisfying a mixing condition in one form or another (cf. ([4],[5])). This becomes clear if one sets $F(x,t,\omega) \equiv F(t,\omega)$.

2. Theorems 1 and 2 can be used, in particular, for a rigorous justification, refinement, and indication of the conditions of applicability of certain conclusions of ([1]). Let us consider, for example, the question of parametric excitation of linear systems by random forces (see ([1]), § 19).

The equation of motion of a system whose frequency undergoes small random perturbations $\varepsilon\xi(t,\omega)$, and whose friction coefficient is γ, has the form

$$\ddot{x} + \mu^2(1+\varepsilon\xi(t,\omega))x + \gamma\dot{x} = 0. \tag{8}$$

If the process $\xi(t,\omega)$ is stationary, ergodic, and has zero mean, with $|\xi(t,\omega)| < C$ with probability 1, and $\gamma = \varepsilon\gamma_1$ ($\gamma_1 = \text{const}$), then, making the change of variables

$$x = e^u\cos(\mu t+\theta), \qquad \dot{x} = -\mu e^u \sin(\mu t+\theta) \tag{9}$$

and applying Theorem 1.1 from ([3]), we obtain that system (8) is stable on a time interval $O(1/\varepsilon)$ for all $\gamma_1 > 0$, and the approximation to the equilibrium can be described by the equation $\ddot{x} + \mu^2 x + \varepsilon\gamma_1\dot{x} = 0$ the more accurately, the smaller ε is.

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A more interesting case is obtained if $\gamma = \varepsilon^2\gamma_1$, where $\gamma_1 = \text{const}$. Then the stability or instability of the system can be detected only on a time interval $O(1/\varepsilon^2)$. If, in addition to the requirements listed above, the process $\xi(t,\omega)$ satisfies condition B2, then Theorem 2 can be applied to the system of equations for the process $\{u_\varepsilon(t,\omega), \theta_\varepsilon(t,\omega)\}$ obtained after the substitution (9) from (8). This theorem makes it possible to establish that on the interval $0 \le \varepsilon^2 t \le \tau_0$, as $\varepsilon \to 0$,

$$\{u_\varepsilon(t,\omega), \theta_\varepsilon(t,\omega)\} \to \{u_0(\varepsilon^2 t,\omega), \theta_0(\varepsilon^2 t,\omega)\},$$

where $u_0(\tau,\omega)$ and $\theta_0(\tau,\omega)$ are mutually independent Markov processes of diffusion type with constant diffusion and drift coefficients. In particular, the process $u_0(\tau,\omega)$ has diffusion coefficient $\mu^2 f(2\mu)/16$ and drift coefficient $\mu^2 f(2\mu) - 4\gamma_1/8$ (here $f(\lambda)$ is the spectral density of the process $\xi(t,\omega)$). Hence it is clear that, if

$$\mu^2 f(2\mu)/4 \ge \gamma_1, \tag{10}$$

system (8) is unstable for sufficiently small ε. If the opposite condition is fulfilled, Theorem 2 permits one to assert only that no loss of stability will occur on a time interval of order $O(1/\varepsilon^2)$. Condition (10) was obtained in ([1]) by means of nonrigorous considerations.

3. An important point in the proof of Theorem 1 is the establishment of the estimate

$$M\left|x_\varepsilon(\tau+h,\omega) - x_\varepsilon(\tau,\omega)\right|^4 < ch^2,$$

which guarantees compactness of the family of distributions associated with the process $x_\varepsilon(\tau,\omega)$ in the space of continuous functions (see (6)). For the establishment of this estimate the following is essential.

Lemma 1. Let $\{\xi_n^{(1)}, \dots, \xi_n^{(2k)}\}$ $(n = 1, 2, \dots)$ be a sequence of random vectors in $E^{(2k)}$ with zero mathematical expectation, satisfying Rosenblatt's strong mixing condition with mixing coefficient $\alpha(\tau)$. Suppose also that either the conditions

$$\left|\xi_n^{(i)}\right| < C, \qquad \int_0^\infty \tau^{k-1}\alpha(\tau)\,d\tau < \infty,$$

or, for some $m > 2$, the conditions

$$M\left|\xi_n^{(i)}\right|^{m(2k-1)} < C, \qquad \int_0^\infty \tau^{k-1}[\alpha(\tau)]^{(m-2)/m}\,d\tau < \infty$$

are fulfilled. Then for all $N > 0$ the estimate

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$$\sum_{n_i=1}^{N} \left| M\left(\xi_{n_1}^{(1)} \cdots \xi_{n_{2k}}^{(2k)}\right) \right| < C_1 N^k. \tag{11}$$

holds.

The proof of this lemma is close in idea to the proof of similar assertions in (7). We note that in the case $\xi_n^{(1)} = \xi_n^{(2)} = \cdots = \xi_n^{(2k)} = \xi_n$, from (11) there follows the estimate

$$M(\xi_1 + \ldots + \xi_N)^{2k} < C_1 N^k,$$

which, as is known, cannot be improved even for independent random variables.

For estimating the conditional moments of first and second order of the increment of the process $x_\varepsilon(\varepsilon^2 t)$, the following lemma, easily following from (4), is useful.

Lemma 2. Let the family of σ-algebras N_s^t satisfy condition (4), let ξ be an N_0^t-measurable random variable, and let $G(x, \omega)$, $|G| < 1$, be a measurable function on $E^l \times \Omega$ which, for each fixed x, is an $N_{t+\tau}^{\infty}$-measurable random variable. Let $g(x) = MG(x, \omega)$. Then with probability 1

$$|M\{G(\xi(\omega), \omega)/N_0^t\} - g(\xi(\omega))| < 2\beta(\tau).$$

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1. R. L. Stratonovich, *Selected Problems in the Theory of Fluctuations in Radio Engineering*, Moscow, 1961.

2. R. L. Stratonovich, *Conditional Markov Processes*, Moscow, 1966.

3. R. Z. Khas' minskii, Theory of Probability and Its Applications, **11**, 2 (1966).

4. I. A. Ibragimov, DAN, **125**, No. 4, 711 (1959).

5. I. A. Ibragimov, Theory of Probability and Its Applications, **7**, 4, 316 (1962).

SovietRxiv

6. Yu. V. Prokhorov, Theory of Probability and Its Applications, **1**, 2, 176 (1956).

7. S. N. Bernstein, Uspekhi Mat. Nauk, vol. **10**, 65 (1944).

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