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Abstract

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MATHEMATICS

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AN EXTENDED CONCEPT OF QUASICONFORMALITY OF A PLANE MAPPING AND LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS OF MIXED TYPE

(Presented by Academician M. A. Lavrent' ev on 30 VI 1965)

In the present note a new definition is given of the concept of quasiconformality of a plane mapping with three characteristics, which in one case passes into the usual definition of quasiconformality with one pair of characteristics, due to M. A. Lavrent' ev. Our general concept is related to the properties of solutions of a linear system of differential equations of mixed type in the same way as ordinary quasiconformality is related to the properties of solutions of the corresponding system of elliptic type.

To bring out the logical side of the construction and to achieve unity of exposition, we use, where necessary, ring terminology. However, the elements of our rings admit a complex notation in the form of double, dual, and ordinary complex numbers, and, in order to emphasize the analogy with the elliptic case, we for the most part give preference to such notation. On the whole, the construction is carried out in accordance with M. A. Lavrent' ev' s scheme for the elliptic case.

1°. **The ring** $K(z;\alpha,\beta,\gamma)$. Fix real numbers $\alpha,\beta,\gamma,\ \gamma\neq 0$. In the set of ordered pairs of real numbers $z=(x,y)$ introduce the operations

$$z_1+z_2=(x_1+x_2,\ y_1+y_2),$$

$$z_1z_2=\left(x_1x_2-\frac{\alpha}{\gamma}y_1y_2,\ x_1y_2+x_2y_1-\frac{2\beta}{\gamma}y_1y_2\right).$$

We obtain a commutative ring $K=K(z;\alpha,\beta,\gamma)$ with unit $(1,0)\equiv 1$. Put $\alpha\gamma-\beta^2=-t$. For $t<0$ the ring K is a field; for $t=0$ it has zero divisors of the form $(c\beta,c\gamma)$, and for $t>0$ of the form $[c(\sqrt{t}\pm\beta)\pm c\gamma]$, c real. In the plane xOy the zero divisors fill a straight line $(t=0)$ or two straight lines $(t>0)$

and determine, respectively, 2 or 4 sectors of the plane. Every element $z \in K$ admits the complex notation

$$z = x - \frac{\beta}{\gamma} y + i_t \frac{1}{\gamma} y,$$

where $i_t = (\beta, \gamma)$, $i_t^2 = (t, 0) = t$.

Introduce the notation

$$\bar{z} = x - \frac{\beta}{\gamma} y - i_t \frac{y}{\gamma}, \qquad |z| = \sqrt{|\bar{z}z|}$$

and define the function $\arg z$, setting it for $t > 0$, $t = 0$, and $t < 0$ equal, respectively, to

$$\frac{1}{2\sqrt{t}} \ln \left| \frac{\gamma x - (\beta - \sqrt{t})y}{\gamma x - (\beta + \sqrt{t})y} \right|, \qquad \frac{y}{\gamma x - \beta y}, \qquad \frac{1}{\sqrt{-t}} \operatorname{arctg} \frac{\sqrt{-t}\, y}{\gamma x - \beta y} + \frac{\varphi}{\sqrt{-t}},$$

where $\varphi = 0, +\pi, -\pi$, and the value of φ is chosen so that, for $\alpha = \gamma = 1$, $\beta = 0$, the value of $\arg z$ coincides with the principal value of the argument of an ordinary complex number.

An element z distinct from a zero divisor is determined by the numbers $|z|$, $\arg z$ and by the choice of a sector of the plane. If z is a proper zero divisor, then $|z| = 0$, $\arg z = \pm\infty$. For $z = 0$ the argument z is not defined.

2°. ***t*-curves and their characteristics.** Let $\gamma \neq 0$, $\alpha + \gamma \geq 0$, $|t| \leq 1$. Consider the curve $|z| = c$ or, $c^2 = \rho h^2 / |\gamma|$ ($\rho > 0$, $h > 0$ are as yet undetermined constants),

$$|\gamma x^2 - 2\beta xy + \alpha y^2| = \rho h^2,$$

i.e.

ellipse ($t < 0$), a pair of parallel straight lines ($t = 0$), or a pair of conjugate hyperbolas ($t > 0$) with asymptotes $\gamma x - (\beta \mp \sqrt{t})y = 0$. We call this geometric image a t-curve with characteristics α, β, γ and denote it by $E_n'(a, \beta, \gamma; z_0)^*$. The angle θ, $0 \leq \theta < \pi$, and the number p such that, after rotation of the axes through the angle θ, the t-curve takes the form

$$\left| -t \frac{x_1^2}{(ph)^2} + \frac{y_1^2}{h^2} \right| = 1,$$

$$h^2 = c^2 |\gamma| / p, \quad p / \sqrt{|t|} \geq 1,$$

are determined by the relations:

$$p = \frac{1}{2}\left(\alpha + \gamma + \sqrt{(\alpha+\gamma)^2 + 4t}\right),$$

$$\operatorname{tg}\theta = \frac{1}{\beta}\left(\gamma - \alpha + \sqrt{(\gamma-\alpha)^2 + 4\beta^2}\right) \quad \text{for } \beta \neq 0;$$

$\theta = 0$ for $\beta = 0$ and $\gamma - \alpha < 0$; $\theta = \pi/2$ for $\beta = 0$ and $\gamma - \alpha > 0$; θ is undefined if $\beta = 0$ and $\gamma = \alpha$;

$$\alpha = p\cos^2\theta - \frac{t}{p}\sin^2\theta; \quad \beta = \left(\frac{t}{p} + p\right)\sin\theta\cos\theta; \quad \gamma = p\sin^2\theta - \frac{t}{p}\cos^2\theta,$$

where to the requirement $\gamma \neq 0$ there corresponds also the condition $t \neq p^2\operatorname{tg}^2\theta$. Thus, a t-curve is specified (up to similarity transformations and parallel translation) by the quantities α, β, γ, $\gamma \neq 0$, $\alpha + \gamma \geq 0$, $\alpha\gamma - \beta^2 = -t$, $|t| \leq 1$, or by the quantities t, p, θ, $|t| \leq 1$, $p \geq \sqrt{|t|}$, $0 \leq \theta < \pi$, $p^2\operatorname{tg}^2\theta \neq t$**.

The numbers t, p, θ we also call the characteristics of the t-curve. If $\alpha = -t$, $\beta = 0$, $\gamma = 1$ (or $p = 1$, $\theta = \pi/2$), then the t-curve

$$|x^2 - ty^2| = h^2$$

is called a t-circle.

3°. Mappings of a ring $K = K(z; \alpha, \beta, \gamma)$ onto a ring $K' = K(w; -t, 0, 1)$. Let $u(x, y), v(x, y)$ be real functions defined in a domain D of the plane xOy. A mapping

$$w = f(z) = u(x, y) + itv(x, y)$$

of the elements

$$z = x - \frac{\beta}{\gamma}y + i_t\frac{y}{\gamma}, \qquad z \in D,$$

of the ring K, $\gamma \neq 0$, $\alpha + \gamma \geq 0$, $\alpha\gamma - \beta^2 = -t$, $|t| \leq 1$, into the ring K' of elements $w = u + itv$ is called a t-mapping with characteristics α, β, γ***.

A sequence of points $\{z_n\}$, $z_n \in K$, t-converges to $z_0 \in K$,

$$z_n \underset{t}{\rightarrow} z_0 \quad \left(\text{or } \lim_{n\to\infty} z_n = z_0\right),$$

if for every $\varepsilon > 0$ and some $\varphi > 0$ there exists N such that for $n > N$,

$$|z_n - z_0| < \varepsilon \quad \text{and} \quad |\arg(z_n - z_0)| < \varphi.$$

A t-mapping $w = f(z)$ of a domain D is called t-continuous at a point $z_0 \in D$ if from $z_n \underset{t}{\to} z_0$ it follows that $f(z_n) \underset{t}{\to} f(z_0)$; a mapping is called t-continuous in D if it is t-continuous at every point of D.

A t-continuous at z_0 t-mapping $f(z)$ is called t-differentiable at this point if the derivatives u_x, u_y, v_x, v_y are finite at z_0 and if, for some $\varepsilon > 0$, in all (ε, φ)-neighborhoods it admits the representation

$$f(z) = f_1(z; z_0) + \varepsilon(z, z_0)(z - z_0),$$

where

$$f_1(z, z_0) \equiv f(z_0) + \frac{\partial f(z_0)}{\partial x}(x - x_0) + \frac{\partial f(z_0)}{\partial y}(y - y_0)$$

is t-continuous at z_0, and

$$\varepsilon(z, z_0) = \varepsilon_1 + i_t \varepsilon_2,$$

$\varepsilon_j(z, z_0)$ $(j = 1, 2)$ are real, with $\varepsilon_j \to 0$ as $z \underset{t}{\to} z_0$****.

Definition 1. $f(z)$ maps an infinitesimal t-curve $E_n(a, \beta, \gamma; z_0)$ into an infinitesimal t-circle, if $f(z)$ is continuous and one-to-one near z_0*****, t-continuous at z_0, and for

* This denotes a t-curve if it has the form $|z - z_0| = \rho h^2/|\gamma|$. In our case $z_0 = 0$.

** $p/\sqrt{|t|}$ is equal to the ratio of the larger semiaxis of the t-curve $(t \neq 0)$ to the smaller.

θ is the angle of inclination of the larger semiaxis to the axis Ox.

For $p = \sqrt{|t|}$ the axes are equal, but the choice of sign before the root in the formula for $\operatorname{tg}\theta$ determines the angle in the case $t > 0$. If $t < 0$, then the condition $p = \sqrt{|t|}$ is equivalent to the condition $\alpha = \gamma$ and $\beta = 0$, and in this case θ is not determined (the t-curve is then specified by the quantities p and t).

*** At the present point the numbers α, β, γ (and hence also the rings K, K') are fixed,

therefore in the following definitions the words "with characteristics α, β, γ" are omitted.

**** For $t < 0$, t-convergence, t-continuity and t-differentiability are equivalent

to the usual notions of convergence, continuity, and differentiability. If $t \geq 0$, however, the notion of t-convergence is narrower than the notion of ordinary convergence; a t-continuous function may be discontinuous in the ordinary sense and conversely.

***** Continuity and one-to-one-ness are understood here as ordinary continuity

and one-to-one-ness of the mapping $u = u(x, y)$, $v = v(x, y)$ in a Euclidean neighborhood of the point z_0.

for every φ, $0 \leq \varphi < \infty$, the condition

$$\lim_{h \to 0} \left[\max_{z \in E_h^{\varphi}} |f(z) - f(z_0)| \Big/ \min_{z \in E_h^{\varphi}} |f(z) - f(z_0)| \right] = 1,$$

is fulfilled, where E_n^{φ} is the set of points $z \in E_n$ for which $|\arg(z - z_0)| \leq \varphi$.

Let $t > 0$. A t-mapping $w = f(z)$, t-continuous at the point z_0, preserves (respectively, changes) the arrangement of pairs of sectors at the point z_0, if, as $z \to z_0$, with z belonging to a definite pair of opposite characteristic sectors with vertex at z_0*, $f(z) \underset{t}{\to} f(z_0)$ in such a way that $f(z)$ belongs to the like-named pair (respectively, to the complementary pair) of characteristic sectors with vertex at $f(z_0)$. A one-to-one mapping, continuous near z_0, preserves (changes) orientation at the point z_0, if a circuit around the point z_0 in a definite direction near z_0 corresponds to a circuit around the point $f(z_0)$ in the same (respectively, in the opposite) direction. A one-to-one mapping, continuous near z_0 and t-continuous at z_0, $w = f(z)$, is called regular at the point z_0, if in the case $\gamma > 0$ it preserves orientation and the arrangement of pairs of sectors at z_0, or changes both, while in the case $\gamma < 0$ it preserves orientation and changes the arrangement of pairs of sectors, or conversely**.

4°. ***t*-quasiconformal mappings.** Let $z = (x, y)$ and $w = (u, v)$. Consider a mapping

$$w = T(z): \quad u = u(x, y), \quad v = v(x, y) \tag{1}$$

of a domain D of the xOy-plane into the uOv-plane. Fix numbers α, β, γ, $\gamma \neq 0$, $\alpha + \gamma \geq 0$, $\alpha\gamma - \beta^2 = -t$, $|t| \leq 1$.

Definition 2. A mapping $w = f(z)$ of the ring $K(z; \alpha, \beta, \gamma)$ onto the ring $K(w; -t, 0, 1)$, constructed from T and α, β, γ as in Sec. 3°, will be called the t-mapping with characteristics α, β, γ corresponding to the mapping (1). We shall say that the mapping (1) possesses the t-property with characteristics α, β, γ (t-continuously, t-differentiably, etc.), if the corresponding t-mapping with these same characteristics has the indicated property.

Definition 3. In the domain D a continuous distribution of characteristics α, β, γ is given if everywhere in D there are defined continuous functions $\alpha(x,y), \beta(x,y), \gamma(x,y)$ such that in D $\gamma(x,y) \neq 0$, $\alpha(x,y) + \gamma(x,y) \geq 0$, $|t(x,y)| \leq 1$, where $t(x,y) = -(\alpha\gamma - \beta^2)$.

In accordance with Sec. 2°, the assignment of a continuous distribution of characteristics α, β, γ is equivalent to the assignment in D of functions $t(x,y)$, $p(x,y)$, and $\theta(x,y)$ such that t is continuous in D and $|t| \leq 1$; $p > 0$ is continuous in D and $p \geq \sqrt{|t|}$; θ is defined everywhere in D where $p \neq \sqrt{-t}$ $(t < 0)$ and satisfies in addition the conditions: $0 \leq \theta < \pi$, $\tan^2\theta \neq t/p^2$, θ is continuous in any closed domain $\overline{D}_1 \subset D$ containing no points at which $p = \sqrt{-t}$ $(t < 0)$. We shall also say of such functions that they define a continuous distribution of characteristics t, p, θ.

Definition 4. Let a continuous distribution of characteristics $t(z), p(z), \theta(z)$, $z = (x,y)$, be given in the domain D. The transformation (1) produces a t-quasiconformal mapping of the domain D with characteristics t, p, θ, if it is regular at every point of D and for any point $z \in D$ transforms an infinitesimal t-curve $E_n[p(z), \theta(z); z]$ into an infinitesimal t-circle.

5°. *t*-quasiconformality and systems of differential equations of mixed type.

Theorem. In order that the continuous mapping (1), with nonzero Jacobian and t-differentiable with characteristics $\alpha(x,y)$, $\beta(x,y)$, $\gamma(x,y)$, be t-quasiconformal with the same characteristics (in the sense of Definition 4)

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* In any ring $K(z; \alpha, \beta, \gamma)$ at the point $z_0 \in K$ there exist two pairs of sectors. The first pair is characterized by the condition $(z - z_0)(\overline{z} - \overline{z_0}) > 0$, the second by the condition $(z - z_0)(\overline{z} - \overline{z_0}) < 0$.

** For $t \leq 0$ the requirement of regularity of the mapping means only preservation of orientation, for $\gamma(z_0) > 0$, and any t-continuous mapping "preserves the arrangement of pairs of sectors," since for $t < 0$ there are no sectors, while for $t = 0$ there is only one pair of sectors.

and, at points where $t(x, y) = 0$, preserve t-angles*, it is necessary and sufficient that it satisfy the system

$$\alpha(x,y)v_x + \beta(x,y)v_y + u_y = 0, \qquad \beta(x,y)v_x + \gamma(x,y)v_y - u_x = 0. \tag{2}$$

6°. Remark. Consider the system

$$a_1u_x + b_1u_y + c_1v_x + d_1v_y = 0, \qquad a_2u_x + b_2u_y + c_2v_x + d_2v_y = 0 \tag{3}$$

with coefficients depending on x, y and continuous in D, at those points where the system does not reduce to a single equation. If at such points

$$a_1d_2 - a_2d_1 = b_1c_2 - b_2c_1, \tag{4}$$

then the system (3) can be reduced by simple transformations to the system (2)**. The study of the local properties of the system (3) without condition (4) will be carried out in another paper.

If $-t = \alpha\gamma - \beta^2 > 0$, then the ring $K(z; \alpha, \beta, \gamma)$ is topologically equivalent to the ring $K(z; 1, 0, 1)$, which makes it possible to pass from a system of local rings (in the case of variable α, β, γ) to a global ring. In this case the connection between the notions of t-quasiconformality and ordinary quasiconformality is easily established (if $t \equiv -1$ in D, then a t-quasiconformal mapping is an ordinary quasiconformal mapping with one pair of characteristics). For $t \geq 0$ and variable α, β, γ, the rings are not topologically equivalent, and the system of local rings can be replaced by a single ring only in the case when α, β, γ are constant in D. Concerning the connections between the notions of t-differentiability, t-quasiconformality, and ordinary differentiability, the author knows only the following: if the mapping (1) is t-differentiable at every point of the domain D with characteristics $\alpha(z)$, $\beta(z)$, $\gamma(z)$, then u and v are differentiable almost everywhere in D in the ordinary sense. Further investigation of these connections is needed.

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[*] $w = f(z)$ preserves t-angles at z_0 if there exists

$$\lim_{\substack{\Delta z \to 0 \\ t}} \arg(\Delta w / \Delta z)$$

or

$$\lim_{\substack{\Delta z \to 0 \\ t}} \arg(\Delta w / \Delta \bar{z}),$$

independent of the manner in which $\Delta z \to 0$. The presence of this requirement is explained by the fact that for $t = 0$ constancy of the "stretchings" does not imply preservation of the "angles."

** The transformation reducing system (3) to system (2), with the required properties of the coefficients α, β, γ, generally speaking exists at each point, but such transformations are ordinary rotations of the coordinate systems xOy and uOv through angles $\pi/4$ or $\pi/1$, and reflections with respect to the axes.

Note: Figure translations are in progress. See original paper for figures.

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