ON THE \(J\)-FUNCTOR OF CELL COMPLEXES

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Abstract

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ON THE J-FUNCTOR OF CELL COMPLEXES

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The J-functor was studied by Rokhlin, Milnor, and Kervaire (^{1,2}) in terms of framed manifolds and the classical J-homomorphism, and also by Atiyah, Adams, and Walker in modern terms ($^{3-7}$). It has been well studied for spheres and projective spaces. Adams, in terms of operations Ψ_Λ^k $(\Lambda = R, C)$, defined the groups $J'(X)$, estimating the order of the groups $\tilde{J}(X)$ from below, and showed that for a pair $X \supset Y$ there is an exact sequence

$$J'(X/Y) \to J'(X) \to J'(Y) \to 0,$$

if the analogous sequence is exact for the K_R-functor. (Recall that in (^{5}) this sequence is written for J'', but in (^{6}) it is proved that $J'' = J'$.) In what follows we shall speak about the complex J-functor $J_c(X) \subset \bar{J}(X)$, whose order differs from the order of $J(X)$ only by a factor of the form 2^n. Recall that it is useful to introduce the functor $\Pi^i(X,Y)$ by means of self-maps of the sphere of degree ± 1, and to consider the J-homomorphism $J : K_c^i(X,Y) \to \Pi^i(X,Y)$, putting $J_c^i(X,Y) = JK_c^i(X,Y)$, with $J_c(X,Y) = J_c^0(X,Y)$. Generally speaking, for the J_c-functor there need not exist an exact sequence of the pair

$$\to J_c^i(X/Y) \xrightarrow{\pi^*} J_c^i(X) \xrightarrow{i^*} J_c^i(Y) \xrightarrow{\delta} J_c^{i+1}(X/Y) \to$$

and we have only that $\operatorname{Ker} i^* \supset \operatorname{Im} \pi^*$, etc.

Define the homology of the J-functor:

$$H_J^i(X,Y) = \operatorname{Ker} i^* / \operatorname{Im} \pi^*, \qquad \operatorname{Ker} i^* \subset J_c^i(X).$$

In what follows we shall be interested in the following groups: a) $H_J^0(X,Y)$, if $X/Y = S^{4n}$; b) $\pi^* J_c^0(X/Y) \subset J_c^0(X)$, $X/Y = S^{4n}$.

1. If the homology of the complex X has no torsion, $X/Y = S^{4n}$, and $H^*(X) = H^*(Y) + H^*(S^{4n})$, then from (^{5,7}) one can extract that there exists an exact sequence

$$Z_2 \to J_c(S^{4n}) \xrightarrow{\pi^*} J_c(X).$$

Consider the commutative diagram

$$\begin{tikzcd}
K_c^0(S^{4n}) \arrow[r, "\pi^*"] \arrow[d, "J"] & K_c^0(X) \arrow[r, "i^*"] \arrow[d, "J"] & K_c^0(Y) \arrow[d, "J"] \\
\Pi^0(S^{4n}) \arrow[r, "\pi^*"] & \Pi^0(X) \arrow[r, "i^*"] & \Pi^0(Y)
\end{tikzcd}$$

and put $B = \mathrm{Ker}(Ji^*) \subset K_c^0(X)$. Let $\alpha \in B$ and $\xi = \alpha + N \in K_c(X)$, where ξ is a vector stable bundle and N is a scalar. Let T_ξ denote the Thom complex of the bundle ξ over X. From the definition of the group $B = \mathrm{Ker}(Ji^*)$ one can extract the following: the complex T_ξ has the homotopy type

$$(S^{2N} \vee E^{2N}Y) \cup_\psi D^{2N+4n},$$

where

$$\psi : S^{2N+4n-1} \to S^{2N} \vee E^{2N}Y,$$

and E is suspension. Since $N \gg n$, we have

$$\psi \in \pi_{2N+4n-1}(S^{2N}) + \pi_{2N+4n-1}(E^{2N}Y),$$

although the decomposition into a direct sum is not unique. Put $X = Y \cup_\varphi D^{4n}$, where $\varphi \in \pi_{4n-1}Y$ and $\psi = \psi_0 + \psi_1$, $\psi_0 \in \pi_{2N+4n-1}(S^{2N})$, $\psi_1 \in \pi_{2N+4n-1}(E^{2N}Y)$.

Lemma 1. $\psi_1 = E^N\varphi, \quad E^N : \pi_{4n-1}(Y) \to \pi_{2N+4n-1}(E^{2N}Y)$.

We do not give the proof of Lemma 1 (see (8), § 7). Let us note that the element ψ_1 is determined uniquely in view of the canonical projection $S^{2N} \vee E^{2N}Y \to E^{2N}Y$, whereas the element ψ_0 is not unique, since the decomposition in $S^{2N} \vee E^{2N}Y$ has a large degree of arbitrariness. However, fixing the decomposition in b induces the fixing of the element ψ_0 and of the projection $\Delta_{\psi_0} : T_\xi \to S^{2N} \cup_{\psi_0} D^{2N+4n}$.

Put $S^{2N} \cup_{\psi_0} D^{2N+4n} = P_{(\xi,\psi_0)}$, $\xi - N \in B$. Let now $\eta \in K_c^0(P_{(\xi,\psi_0)})$ be such an element that $\mathrm{ch}\,\eta = U + \lambda V$, where U, V are generators of the groups $H^{2N}(P_{(\xi,\psi_0)}, Z) = H^{2N+4n}(P_{(\xi,\psi_0)}, Z) = Z$ and $\lambda \in Q$, Q is the rational numbers. We denote $\lambda \bmod 1 \in Q/Z$ by $\lambda(\xi, \psi_0) = \lambda$.

The following substantial, though simple, lemma holds.

Lemma 2. *The number* $\lambda(\xi, \psi_0) \in Q/Z$ *depends only on the element* $a = \xi - N \in K_c^0(X)$ *such that* $J \cdot i^*(a) = 0$. *Moreover, the correspondence*

$$a \mapsto \lambda(a + N, \psi_0)$$

induces a single-valued homomorphism $B \to Q/Z$, *where* $B = \operatorname{Ker}(J \cdot i)$.

Proof. It is easy to see that $\lambda(\xi, \psi_0)$ can depend only on the ambiguity in the choice of $\psi_0 \in \Pi(S^{4n}) = \pi_{2N+4n-1}(S^{2N})$, which also generates the kernel of the mapping $\pi^* : \Pi(S^{4n}) \to \Pi(X)$. Let $\psi_0^{(1)}$, $\psi_0^{(2)} \in \Pi(S^{4n})$ be constructed from ξ, where $\xi = a + N$, $a \in B$. We have two projections

$$\Delta_{\psi_0^{(1)}} : T_\xi \to P_{\left(\xi, \psi_0^{(1)}\right)}, \qquad \Delta_{\psi_0^{(2)}} : T_\xi \to P_{\left(\xi, \psi_0^{(2)}\right)}.$$

In cohomology we see that the element $\bar{V} = \Delta^*_{\psi_0^{(1)}} V \in H^{4n}(X, Z)$ is determined uniquely, independently of the choice of ψ_0. Obviously, we have

$$\operatorname{ch}(\Delta_{\psi_0^{(1)}} \eta_1) - \operatorname{ch}(\Delta_{\psi_0^{(2)}} \eta_2) = (\lambda_1 - \lambda_2)\bar{V},$$

where $\eta_i \in K_c^0(P_{(\xi, \psi_0^{(i)})})$, $i = 1, 2$, are elements such that $\operatorname{ch} \eta_i = U + \lambda_i V$.

Since the element

$$\Delta_{\psi_0^{(1)}} \eta_1 - \Delta_{\psi_0^{(2)}} \eta_2 \in K_c^0(X)$$

has filtration $\geq 2n = \dim \bar{V}/2$, we see from complex Bott periodicity that $\lambda_1 - \lambda_2$ is an integer. Therefore the numbers $\lambda(\xi, \psi_0^{(1)})$ and $\lambda(\xi, \psi_0^{(2)}) \in Q/Z$ coincide. Thus the mapping $B \to Q/Z$, defined by $\lambda(\xi, \psi_0)$, is well defined. It is not difficult to show that it is additive. The lemma is proved.

From the construction of the mapping $\lambda(\xi, \psi_0) = \lambda(\xi) = \lambda(a)$, $a \in B$, $\xi = a + N$, it is clear that this mapping $\lambda : B \to Q/Z$ gives rise to a mapping $\lambda : J(B) \to Q/Z$, where $J(B) \in \Pi(X)$, and $J(B) = \operatorname{Ker} i^* \subset J_c(X)$. Consider in the group B the subgroup $B' = \pi^* K_c^0(S^{4n})$. We have the composition $\lambda \cdot \pi^* : K_c^0(S^{4n}) \to Q/Z$.

Lemma 3. *The mapping* $\lambda \cdot \pi^* : K_c^0(S^{4n}) \to Q/Z$ *decomposes into the composition* $\lambda \cdot \pi^* = \bar{\lambda} \cdot J$

$$K_c^0(S^{4n}) \xrightarrow{J} \Pi^0(S^{4n}) \xrightarrow{\bar{\lambda}} Q/Z,$$

and on the group $J_c(S^{4n}) = JK_c^0(S^{4n})$ *the kernel of the mapping* $\bar{\lambda}$ *contains no more than two elements. Moreover, the mapping* $\bar{\lambda}$ *decomposes into the composition* $\bar{\lambda} = \bar{\lambda}' \cdot \pi^*$

$$\Pi^0(S^{4n}) \xrightarrow{\pi^*} \pi^*\Pi^0(S^{4n}) \xrightarrow{\bar{\lambda}'} Q/Z.$$

For the proof, let us note that the mapping $\lambda\pi^*$ is obviously well defined on the group $J_c(S^{4n}) = JK_c^0(S^{4n})$. The mapping $\bar{\lambda} : \Pi^0(S^{4n}) \to Q/Z$ is defined through the two-cell complex $S^{2N} \cup_h D^{2N+4n}$ by analogy with $\lambda(\xi, \psi_0)$ for fixed ψ_0, and, as is easy to see, the mapping $\overline{\lambda}$ is trivial on the kernel $\operatorname{Ker} \pi^*$, since $\lambda\pi^* = \overline{\lambda}J$. The rest follows easily from comparing this with the results of Milnor, Kervaire (^{2}), and Adams (^{4,5}).

Remark. For two-cell complexes the mapping $\overline{\lambda}$ was, apparently, defined by Adams in connection with the question of splitting off $J(S^{4n})$ in $\Pi(S^{4n})$ as a direct summand in a work not yet published (see (^{6}), introduction).

Theorem 1. The homomorphism

$$\overline{\lambda} : \Pi(S^{4n}) \to Q/Z$$

annihilates all compositional elements $\alpha \cdot \beta$, where $\alpha \in \Pi(S^j)$, $\beta \in \Pi(S^k)$, $j + k = 4n + 1$, $j \geqslant 2$, $k \geqslant 2$, $\Pi(S^q) = \pi_{N+q-1}(S^N)$. In particular, the compositional elements in $J(S^{4n})$ form a subgroup of order at most Z_2. For $n = 2l$, the existence in $J(S^{4n})$ of a nontrivial compositional element is equivalent to

$$|J(S^{4n})| = 2 \cdot (\text{denominator } B_k/4k),$$

where B_k is the Bernoulli number with index k.

The proof follows from the construction and properties of the homomorphism $\overline{\lambda}$ and the results of (^{5}).

From the construction of the homomorphism $\lambda(\xi, \psi_0)$ it is clear that the “denominator” of its image in Q/Z coincides with the order of the group $J_c(S^{4n})$ up to a factor of 2. Comparing this with the definition of the group $B = \operatorname{Ker}(Ji^*) = \operatorname{Ker}(i^*J) \subset K_c^0(X)$ and with the fact that the homomorphism $\lambda : B \to Q/Z$ depends only on $JB = \operatorname{Ker} i^* \subset J_c(X)$, we obtain the following assertion.

Theorem 2. If the group $\operatorname{Ker} \pi^*$ is Z_2, where

$$\pi^* : J_c(S^{4n}) \to J(B) \subset J_c(X),$$

then there is a direct decomposition

$$J(B) = \operatorname{Im} \pi^*(J_c(S^{4n})) + H_J^0(X, Y), \qquad H_J^0(X, Y) = J(B)/\pi^* J_c(X, Y).$$

In all cases such a direct decomposition holds for the quotient group B/Z_2.

II. Let us now dwell in somewhat greater detail on the case when

$$X = S^{4l} \cup_\varphi D^{4(k+l)}, \qquad k < l.$$

Here, evidently, $Y = S^{4l}$ and $X/Y = S^{4(k+l)}$. For stable two-cell complexes we have the invariant $\overline{\lambda}(\varphi) \in Q/Z$, where for some $\eta \in K_c^0(X)$ we have

$$\operatorname{ch} \eta = a + \overline{\lambda}(\varphi) b;$$

a, b are generators respectively of the groups

$$H^{4l}(X, Z) = H^{4(k+l)}(X, Z) = Z.$$

Let $\alpha \in B \subset K_c^0(X)$. Consider the Thom complex T_ξ for $\xi = \alpha + N$. Then we have, as is easy to show, an element $\mu \in K_c^0(T_\xi)$ such that

$$\operatorname{ch} \mu = A + nB + \delta(n, \xi, \mu)C,$$

where A, B, C are respectively generators of the groups

$$H^{2N}(T_\xi, Z) = H^{2N+4l}(T_\xi, Z) = H^{2N+4(k+l)}(T_\xi, Z) = Z,$$

and $\delta(n, \xi, \mu)$ is a rational number, n an integer. On the other hand, we previously constructed, with the help of ψ_0, the invariant $\lambda(\xi, \psi_0)$, which defines the mapping $\lambda : B \to Q/Z$.

There is the simple

Lemma 4. The number $\delta(n, \xi, \mu)$ mod 1 depends only on n and on ξ.

Using the previously proved Lemma 1, one can easily obtain the following assertion:

Lemma 5. The equality

$$\delta(n, \xi, \mu) = n\overline{\lambda}(\varphi) + \lambda(\xi, \psi_0) \pmod 1$$

holds.

This lemma follows easily from the fact that, in essence, we defined $\lambda(\xi, \psi_0)$ on B in the same way as $\delta(n, \xi, \mu)$, but taking $n = 0$. The rest follows from this by Lemma 1.

Let B_k be the k-th Bernoulli number and

$$B_k/2k = C_k/D_k,$$

where C_k and D_k are relatively prime.

We indicate the following facts: a) the denominator of $\overline{\lambda}(\varphi)$ is $2D_l$ or a divisor of it; b) the denominator of $\lambda(\xi, \psi_0)$ is $2D_{k+l}$ or a divisor of it.

From this we see that

$$2D_{k+l}(\delta(n, \xi, \mu) - n\overline{\lambda}(\varphi)) \equiv 0 \pmod 1.$$

Let us now carry out the calculation in the following situation: $\alpha \in K_c(S^{4k})$, $\operatorname{ch} \alpha = 2l + u$, where u is a basis element of the group $H^{4l}(S^{4l}, Z) = Z$, l is a scalar,

$$X = T_\alpha = S^{4l} \cup_\varphi D^{4(k+l)}.$$

We have the Thom isomorphisms

$$\varphi_H : \ H^j(S^{4k}) \to H^{4l+j}(T_\alpha), \quad j \geqslant 0; \quad \varphi_K : \ K_c(S^{4k}) \to K_c^0(T_\alpha),$$

where the elements $\varphi_H(1)$ and $\varphi_H(u)$ are denoted by a and b. We choose the isomorphism φ_K so that $\operatorname{ch} \varphi_K(1) = \varphi_H(T(\alpha))$, where $T(\alpha)$ is the Todd genus. Obviously,

$$T(\alpha) = 1 + \frac{B_k}{2k} u.$$

Therefore

$$\operatorname{ch} \varphi_K(1) = a + \frac{B_k}{2k} b = a + \frac{C_k}{D_k} b.$$

It is easy to see that $C_k / D_k = \bar{\lambda}(\varphi) \bmod 1$.

From the results of Adams$^{(5)}$ we extract that for the element $\bar{\eta} = \varphi_K(1) \in K_c^0(T_\alpha) = K_c^0(X)$, $\eta = 2a_{lD} l \bar{\eta}$ lies in $B = \operatorname{Ker} J \cdot i^*$, where $a_l = 1$ if $l = 2l' + 1$; $a_l = 2$ if $l = 2l'$.

Obviously,

$$\operatorname{ch}(2a_{lD} l \bar{\eta}) = 2a_{lD} la + \frac{2a_{lD_{lC}}k}{D_k} b.$$

Consider the stable bundle $\xi = N + \eta \in K_c(X)$. Then, as is easy to show,

$$T(\xi) = 1 + (-1)^l 2a_{lD} l \left[\frac{C_l}{D_l} a + \frac{C_{k+l} C_k}{D_{k+l} D_k} b \right].$$

For the Thom complex T_ξ of the bundle ξ we shall have

$$\operatorname{ch} \varphi_K(1) = \varphi_{HT}(\xi).$$

Further, from what has been said the following equalities follow:

$$\bar{\lambda}(\varphi) = C_k / D_k, \qquad n = (-1)^l 2a_{lC} l, \qquad \mu = \varphi_K(1),$$

$$\delta(n, \xi, \mu) = (-1)^l 2a_{lC_{k+l}} C_{kD} l / D_{k+l} D_k.$$

Since

$$(\delta(n, \xi, \mu) - n\bar{\lambda}(\varphi)) D_{k+l} \equiv 0 \bmod 1,$$

we have

$$2^2 a_{lC} k \left(\frac{C_{k+l} D_l - C_{l D_{k+l}}}{D_k} \right) \equiv 0 \bmod 1, \qquad l > k.$$

Thus the following assertion is obtained:

$$2^2 a_l \frac{C_{k+l} D_l - C_{l D_{k+l}}}{D_k} \equiv 0 \bmod 1, \qquad l > k,$$

where $B_k/2k = C_k/D_k$ and C_k, D_k are relatively prime.

This relation for Bernoulli numbers was obtained in a strange way from topological considerations.

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REFERENCES

1. V. A. Rokhlin, DAN, 81, 19 (1951).
2. J. Milnor, M. A. Kervaire, Proc. Intern. Congress Math. Edinburgh, 1958, p. 454, Cambridge, 1960.
3. M. F. Atiyah, Proc. London Math. Soc. (3), 11, 291 (1961).
4. J. F. Adams, Topology, 2, 181 (1963).
5. J. F. Adams, Topology, 3, 137 (1965).
6. J. F. Adams, Topology, 3, 193 (1965).
7. J. F. Adams, G. Walker, Proc. Cambr. Phil. Soc., 61, 81 (1965).
8. S. P. Novikov, Izv. AN SSSR, ser. matem., 28, 365 (1964).

Note: Figure translations are in progress. See original paper for figures.