A NEW CLASS OF FINITE-DIFFERENCE OPERATORS

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Abstract

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MATHEMATICS

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A NEW CLASS OF FINITE-DIFFERENCE OPERATORS

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It is known that in the class of finite-difference methods for integrating systems of ordinary differential equations using one or several formulas of the form

$$L_n y_k \equiv \sum_{i=0}^{n} a_i y_{k-i} - h \sum_{i=0}^{n} b_i y'_{k-i} = 0, \tag{1}$$

there are no stable methods of degree $p > n + 2$ (see (1)). In the present paper a new class of operators $L_{n\nu}(\alpha)$, depending on the real parameter α, is studied; it contains the ordinary operators L_n for $\alpha = 0$ and $\nu = n$. The extension of the class of operators makes it possible to study finite-difference integration methods more deeply and to construct stable methods of higher degree (in particular, methods of degree $2n$).

1. Consider the family of operators

$$L_{n\nu}(\alpha) = \sum_{i=0}^{\nu} a_i(\alpha) T^{-i} - h \sum_{i=0}^{n} b_i(\alpha) T^{-i-\alpha} D, \tag{2}$$

where D is the differentiation operator: $Dy(x) = y'(x)$; T^s is the shift operator by sh: $T^s y(x) = y(x+sh)$; $a_i(\alpha)$ and $b_i(\alpha)$ are real functions of the parameter α, defined on the entire number axis with the exception of at most a finite number of points and satisfying the conditions

$$a_0(\alpha) \equiv 1, \qquad a_\nu(\alpha) \neq 0, \qquad b_0(\alpha) \neq 0, \qquad b_n(\alpha) \neq 0.$$

Fixing the value of the parameter α, we single out from the family (2) a particular operator. By the **degree of the operator** $L_{n\nu}(\alpha)$ from the family (2) we shall mean the greatest integer $p(\alpha)$ such that

$$L_{n\nu}(\alpha) x^r \equiv 0, \qquad r = 0, 1, \dots, p(\alpha).$$

The number $p = \min_\alpha p(\alpha)$ will be called the **degree of the family of operators** $L_{n\nu}(\alpha)$. We shall be interested only in families of degree $p \geq n+1$. It can be proved that such families are uniquely determined by specifying the functions $a_0(\alpha), \dots, a_\nu(\alpha)$ and can be represented in the form

$$L_{n\nu}(\alpha) = \sum_{i=1}^{\nu} A_i(\alpha)(E - T^{-1})^i - h \sum_{i=0}^{n} B_i(\alpha)(E - T^{-1})^i T^{-\alpha} D, \tag{3}$$

where

$$A_i(\alpha) = (-1)^i \sum_{j=i}^{n} C_j^i a_j(\alpha), \qquad i = 1, \dots, \nu,$$

$$B_i(\alpha) = (-1)^i \sum_{j=i}^{n} C_j^i b_j(\alpha), \qquad i = 0, 1, \dots, n.$$

The simplest family in the class of operators under consideration is the family of operators $L_{n1}(\alpha)$ of degree $n+1$

$$L_{n1}(\alpha) \equiv L_n(\alpha) = E - T^{-1} - h \sum_{i=0}^{n} P_i(\alpha)(E - T^{-1})^{iT^{-\alpha}} D,$$

where

$$P_i(\alpha) = \frac{1}{i!} \int_{\alpha-1}^{\alpha} (t)_i \, dt, \qquad i = 0, 1, \dots,$$

$$(t)_i = t(t+1) \dots (t+i-1), \qquad i = 1, 2, \dots, \qquad (t)_0 \equiv 1.$$

The operators $L_n(\alpha)$, which we shall call **elementary operators**, can be used to construct general finite-difference operators $L_{n\nu}(\alpha)$ of degree $p \geq n+1$. Namely, the following holds.

Theorem 1. *Every family of operators* $L_{n\nu}(\alpha)$ *of degree* $p \geq n+1$ *is uniquely representable in the form of a linear combination of elementary operators:*

$$L_{n\nu}(\alpha) = \sum_{j=0}^{\nu-1} \widetilde{A}_j(\alpha) T^{-j} L_n(\alpha - j),$$

where

$$\widetilde{A}_i(\alpha) = (-1)^i \sum_{k=i+1}^{\nu} C_{k-1}^i A_k(\alpha).$$

This theorem proves to be very useful in the study of general finite-difference operators. Using it, one can prove that if the vector function $y(x)$ has $n+q+2$ continuous derivatives, then

$$L_{n\nu}(\alpha)y(x) = \sum_{i=1}^{q} B_{n+i} T^{-\alpha} \nabla^i y'(x) + h \int_{\alpha-\nu}^{\alpha} \widetilde{A}_{[-\alpha-t]}(\alpha) R_{n+q}(t)\, dt, \qquad (4)$$

where $R_{n+q}(t) = O(h^{n+q+1})$ is the remainder term of Newton' s interpolation formula; $\nabla y(x) \equiv y(x) - y(x-h), \dots$. It follows from (4) that, in order for the family of operators $L_{n\nu}(\alpha)$ to have degree $p = n+q$, it is necessary and sufficient that

$$B_{n+j}(\alpha) \equiv \sum_{i=0}^{\nu-1} A_{i+1}(\alpha) P_{n+j-i}(\alpha) = 0, \qquad j = 1, \dots, q-1. \qquad (5)$$

For any $q \leq \nu$, the system (5) has a nontrivial solution; for $q = \nu$ this solution is unique. From what has been set forth it follows:

Theorem 2. *There exists a unique family of operators $L_{n\nu}(\alpha)$ of degree $n+\nu$. The degree $n+\nu$ is the greatest possible degree for families of operators of the class under consideration.*

An operator $L_{n\nu}(\alpha)$ from a family of degree p will be called **stable** if its corresponding characteristic equation

$$\sum_{i=0}^{\nu} a_i(\alpha) z^{\nu-i} = 0$$

satisfies the stability condition (see (1)). The set of values α for which the corresponding operators from the family under consideration are stable will be called the **region of stability** of the family of operators $L_{n\nu}(\alpha)$ and denoted by $\mathfrak{A}$.

It is easy to verify that, for stability of the operator $L_{n\nu}(\alpha)$, it is necessary and sufficient that the roots of the equation

$$\sum_{i=1}^{\nu} A_i(\alpha)(1-\zeta)^{i-1} = 0 \qquad (6)$$

satisfy the conditions: $|\zeta_i| \geq 1$, and if $|\zeta_i| = 1$, then ζ_i is a simple root not equal to one. If the degree of the family under consideration is $p = n + \nu$, then the left-hand side of equation (6) is the determinant of the matrix

$$P(\alpha, \zeta) = \begin{pmatrix} P_{n+1}(\alpha) & P_n(\alpha) & \cdots & P_{n+2-\nu}(\alpha) \\ \cdots & \cdots & \cdots & \cdots \\ P_{n+\nu-1}(\alpha) & P_{n+\nu-2}(\alpha) & \cdots & P_n(\alpha) \\ 1 & (1-\zeta) & \cdots & (1-\zeta)^{\nu-1} \end{pmatrix},$$

and for sufficiently large values of α, equation (6) is equivalent to the equation

$$\sum_{k=0}^{\nu-1} (-1)^k \left\{ C_{\nu-1}^k + O\left(\frac{1}{\alpha}\right) \right\} \zeta^k = 0.$$

Investigation of the behavior of the roots of this equation as $\alpha \to \infty$ leads us to the following theorem.

Theorem 3. For any $\nu \geq 1$ there exists a number $n_0 = n_0(\nu)$ and a value of the parameter $\alpha_0 = \alpha_0(\nu)$ such that the stability region of the family of operators $L_{n\nu}(\alpha)$, $n \geq n_0$, of degree $n + \nu$ contains the interval $(-\infty, \alpha_0]$.

The value $n_0 = n_0(\nu)$ is comparatively easy to compute for small values of ν, and it turns out that $n_0 < \nu$. Thus, $n_0(1) = \cdots = n_0(5) = 0$; $n_0(6) = 1$; $n_0(7) = 2$. Hence, at least for $n \leq 7$, we can assert that in the class of operators under consideration there exists an infinite number of stable operators of degree $2n$. The interval $(-\infty, \alpha_0]$ does not exhaust the entire stability region. Thus, for example, when $\nu = 1$ the stability region is the entire real axis; the stability region of the families of operators $L_{n2}(\alpha)$ of degree $n + 2$ (for any $n \geq 0$) is the collection of intervals

$$\mathfrak{A}_{n,-1} = (-\infty, \alpha_{n,n-1}), \qquad \mathfrak{A}_j = [\alpha_j^*, \alpha_{n,j}), \qquad j = 0, 1, \ldots, n-2,$$

where α_j^* are the roots of the equation $2P_{n+1}(\alpha) - P_n(\alpha) = 0$, and α_{nj} are the roots of the equation $P_n(\alpha) = 0$, with $\alpha_{n0} > 0$.

2. Consider predictor-corrector methods using, as corrector formulas,

$$L_{n\nu}(\alpha) y_k \equiv \sum_{i=0}^{\nu} a_i(\alpha) y_{k-i} - h \sum_{i=0}^{n} b_i(\alpha) y'_{k-\alpha-i} = 0, \tag{7}$$

where $L_{n\nu}(\alpha)$ is a stable operator from the family of degree $n+\nu$. As predictors, take, for example, the formulas

$$L^*_{n\nu}(\alpha) y_k \equiv y_{k-\alpha} + \sum_{i=1}^{\nu} a_i^*(\alpha) y_{k-i} - h \sum_{i=0}^{n} b_i^*(\alpha) y'_{k-i-\alpha-1} = 0, \tag{8}$$

where $L^*_{n\nu}(\alpha)$ is an operator of degree $n+\nu$. The error of the method (7)–(8) in integrating the system of equations

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

on any finite interval has order $O(h^{n+\nu})$ (assuming sufficient smoothness of the solution of the integrated system). For it, by the usual method (see (2)) one can obtain an asymptotic expansion of the form

$$d_k = \varkappa(\alpha) h^{n+\nu} \int_0^{x_k} \Omega(x, \xi) y^{(n+\nu+1)}(\xi)\, d\xi + O(h^{n+\nu+1}),$$

where $\Omega(x, \xi)$ is a bounded matrix independent of α and h. Minimizing the coefficient

$$\varkappa(\alpha) = B_{n+\nu}(\alpha) \left[\sum_{i=0}^{n} b_i(\alpha) \right]^{-1}$$

over the domain $\mathfrak{A}$ (which presents no difficulty, since $B_{n+\nu}(\alpha)$ and $b_i(\alpha)$ are polynomials), we single out from the collection of methods (7)–(8) the optimal methods. It may turn out that in the domain $\mathfrak{A}$ there are zeros of the polynomial $B_{n+\nu}(\alpha)$. Then the optimal methods will have degree $n+\nu+1$. Thus, for example, if $\nu = 1$, then $\varkappa(\alpha) = P_{n+1}(\alpha)$, and from the easily verified relations

$$\operatorname{sign} P_{n+1}(j) = (-1)^j, \qquad j = 1, 0, -1, \dots, -n+1,$$

$$P'_{n+1}(\alpha) = (\alpha)_n (n!)^{-1},$$

it follows that for any $n \geq 0$ there exist $n+1$ optimal methods of degree $n+2$, corresponding to the zeros of the polynomial $P_{n+1}(\alpha)$. In the class of methods under consideration, the ordinary Adams method, corresponding to $\alpha = 0$ for $\nu = 1$, cannot be optimal, since the coefficient $\varkappa(\alpha) = P_{n+1}(\alpha)$ has a nonzero local extremum at $\alpha = 0$.

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Note: Figure translations are in progress. See original paper for figures.

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