ON SEMI-ISOMORPHISMS AND STRUCTURAL ISOMORPHISMS OF CANCELLATIVE SEMIGROUPS

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Abstract

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ON SEMI-ISOMORPHISMS AND STRUCTURAL ISOMORPHISMS OF CANCELLATIVE SEMIGROUPS

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As in ([1]), by a **semi-isomorphism** of a semigroup Γ onto a semigroup Γ' we mean a one-to-one mapping φ of Γ onto Γ' such that, for any $x, y \in \Gamma$, at least one of the equalities

$$\varphi(xy) = \varphi(x)\varphi(y), \qquad \varphi(xy) = \varphi(y)\varphi(x)$$

holds. In ([1]) Scott proved that every semi-isomorphism of a cancellative semigroup Γ onto a cancellative semigroup Γ' is an isomorphism or an anti-isomorphism. In the proof, the validity of the cancellation law in the semigroup Γ' was used essentially; therefore the question naturally arose whether this requirement could be dropped. It turns out that it can, and thus Scott' s result just mentioned can be considerably strengthened. Namely, the following is true.

Theorem 1. *Every semi-isomorphism of a cancellative semigroup onto an arbitrary semigroup is an isomorphism or an anti-isomorphism.*

Let us note that, as simple examples show (one of them is given in ([1])), Theorem 1 can no longer be extended to arbitrary semigroups.

Theorem 1, which is also of some independent interest, can also be used in the study of structural isomorphisms of cancellative semigroups. The point is that the consideration of structural isomorphisms of semigroups often leads to semi-isomorphisms: sometimes it turns out that a given structural isomorphism of a semigroup induces its semi-isomorphism. In various cases it has been proved separately for cancellative semigroups—taking into account the specific nature of the semigroups under study—that this mapping will be an isomorphism or an anti-isomorphism (see, for example, ([2],[3])). As Theorem 1 shows, the specific nature of the semigroups considered in the final parts of the corresponding proofs need not have been taken into account.

We indicate one new result that can be obtained using Theorem 1. It concerns positively orderable semigroups. Recall that an element a of an ordered* semigroup Γ is called **positive** if $ax \geq x$ and $xa \geq x$ for every $x \in \Gamma$. The dual notion is that of a negative element. A semigroup is called **positively orderable** if it can be ordered in such a way that all its elements are positive. It is easy to see that in a positively orderable cancellative semigroup either all elements have infinite order, or the only element of finite order is an externally adjoined identity.

Theorem 2. *An arbitrary semigroup Γ' is structurally isomorphic to a positively orderable cancellative semigroup Γ if and only if it is either isomorphic or anti-isomorphic to Γ*

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* Here and everywhere below, linear order is meant.

(and in this case every structural isomorphism of Γ onto Γ' is induced by an isomorphism or an anti-isomorphism), or, in the contrary case, is obtained from Γ by replacing the identity by an externally adjoined zero, and the subsemigroups of all nonidentity elements of Γ by an isomorphic or anti-isomorphic one (in this case every structural isomorphism of Γ onto Γ' is induced by a mapping that is an isomorphism or an anti-isomorphism on the subsemigroup of nonidentity elements of Γ and carries the identity of Γ to the zero of Γ').

We give a number of consequences of Theorem 2.

Corollary 1. A semigroup structurally isomorphic to a positively ordered semigroup with the cancellation law is itself positively ordered.

Corollary 2. In order that every structural isomorphism of a positively ordered semigroup with the cancellation law Γ be induced by an isomorphism or an anti-isomorphism, it is necessary and sufficient that Γ have no identity.

Corollary 3. Every structural isomorphism of a positively ordered semigroup with the cancellation law onto a semigroup without zero is induced by an isomorphism or an anti-isomorphism.

The following two assertions are special cases of Corollary 3 (the second of them also follows from Corollary 2), since the semigroups indicated in them are positively ordered.

Corollary 3′. Every structural isomorphism of a goloid (see ([4])) semigroup with the cancellation law onto a semigroup without zero is induced by an isomorphism or an anti-isomorphism.

Corollary 3″ (Petropavlovskaya' s theorem ([2,4])). Every structural isomorphism of a free semigroup is induced by an isomorphism or an anti-isomorphism.

It is known (see ([4])) that in a semigroup with the cancellation law Γ the set of all noninvertible elements forms a subsemigroup (in particular, the empty one when Γ is a group), which, obviously, is an ideal. Taking this into account, applying Theorem 8 and Theorem 1, one can prove

Corollary 4. Let Γ be an ordered semigroup with the cancellation law and with identity e, let H be the subsemigroup of all its noninvertible elements, and suppose that every structural isomorphism of H is induced by an isomorphism or an anti-isomorphism. If e is externally adjoined in Γ, then the assertion of Theorem 2 is true for Γ. If e is not externally adjoined, then every structural isomorphism of Γ is induced by an isomorphism or an anti-isomorphism.

A special case of Corollary 4 (for $H = \emptyset$) is

Corollary $4'$ (Kutyev' s theorem ([5,6])). If all elements of an ordered semigroup Γ are invertible, i.e. Γ is an ordered group, then every structural isomorphism of Γ is induced by an isomorphism or an anti-isomorphism.

From Corollary 4, in view of Corollary 2, the following assertion also follows, more general than the preceding one:

Corollary $4''$**.** If a semigroup with the cancellation law Γ contains nonidentity invertible elements and can be ordered in such a way that every positive (negative) element of it is invertible, then every structural isomorphism of Γ is induced by an isomorphism or an anti-isomorphism.

It is natural to raise the question of a possible extension of Theorem 2 to arbitrary ordered semigroups with the cancellation law. Corollary 4 completely reduces this question to the case of semigroups without invertible elements, i.e. equivalently, semigroups without identity. Indeed, from the indicated corollary it follows directly

Corollary $4'''$**.** Theorem 2 will be valid for an arbitrary ordered cancellative semigroup if and only if every structural isomorphism of any ordered cancellative semigroup without identity is induced by an isomorphism or an anti-isomorphism.

By virtue of Corollary $4''$ and Theorem 1, the question mentioned reduces to the following: **will every structural isomorphism of an ordered cancellative semigroup without identity be induced by a semi-isomorphism?** Of course, in order to resolve this question it suffices to consider only semigroups with two generators.

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Note: Figure translations are in progress. See original paper for figures.