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# COEFFICIENT-FREE EQUATIONS IN WORDS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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**COEFFICIENT-FREE EQUATIONS IN WORDS**

*(Presented by Academician P. S. Novikov on 18 II 1966)*

Let  $k > 1$ ,  $n > 1$ , and let  $A = \{a_1, \dots, a_k\}$ ,  $X = \{x_1, \dots, x_n\}$  be disjoint alphabets. A system of equalities

$$\begin{aligned} \Phi_1 &= \Psi_1 \\ \cdot & \cdot \cdot \cdot \\ \Phi_N &= \Psi_N \quad (N \geq 1) \end{aligned} \tag{*}$$

where  $\Phi_1, \dots, \Phi_N, \Psi_1, \dots, \Psi_N$  are words in the alphabet  $X$ , will be called a **system of coefficient-free equations in words with  $n$  unknowns**. A list  $(X_1, \dots, X_n)$  of words in the alphabet  $A$  will be called a solution of the system (\*) if, in the alphabet  $A$ , the equalities

$$S_{X_1, \dots, X_n}^{x_1, \dots, x_n} \Phi_i = S_{X_1, \dots, X_n}^{x_1, \dots, x_n} \Psi_i \quad (i = 1, \dots, N),$$

hold, where by

$$S_{X_1, \dots, X_n}^{x_1, \dots, x_n} \Phi$$

is denoted the result of substituting in the word  $\Phi$ , in place of each occurrence in  $\Phi$  of the letter  $x_i$ , the word  $X_i$  ( $i = 1, \dots, n$ ). A solution  $(X_1, \dots, X_n)$  of the system (\*) will be called **trivial** if there exist a word  $S$  and natural numbers  $a_1, \dots, a_n$  such that  $X_i = S^{a_i}$  ( $i = 1, \dots, n$ ).

The search for trivial solutions of the system (\*) is evidently reduced to the search for natural solutions of a system of linear Diophantine equations. Therefore it is the search for nontrivial solutions of the system (\*) that is of interest.

Let  $\mathfrak{A}, \mathfrak{B}$  be systems of coefficient-free equations in words with  $n$  unknowns. We shall call the systems  $\mathfrak{A}, \mathfrak{B}$  **equivalent with respect to nontrivial solutions** if every nontrivial solution of the system  $\mathfrak{A}$  is a solution of the system  $\mathfrak{B}$ , and every nontrivial solution of the system  $\mathfrak{B}$  is a solution of the system  $\mathfrak{A}$ .

**Theorem 1.** *The system of equations (\*) is equivalent with respect to nontrivial solutions to the equation*

$$\Phi_1(\Omega)^2\Phi_2 \dots (\Omega)^2\Phi_N = \Psi_1(\Omega)^2\Psi_2 \dots (\Omega)^2\Psi_N,$$

where

$$\begin{aligned} \Omega &= ZUZV; \quad Z = (x_1 \dots x_n)^{5M}; \quad M = \max\{[\Phi_1^\partial, \dots, [\Phi_N^\partial, [\Psi_1^\partial, \dots, \\ &\dots, [\Psi_N^\partial, n]; \quad U = U_1 \dots U_n; \quad V = V_1 \dots V_n; \quad U_i = x_1 x_i x_2 x_i \dots x_n x_i \\ &(i = 1, \dots, n); \quad V_i = x_i x_1 x_i x_2 \dots x_i x_n \quad (i = 1, \dots, n). \end{aligned}$$

This theorem, analogous to Theorem 2 of <sup>(1)</sup>, makes it possible, in searching for nontrivial solutions of the system (\*), to restrict oneself to the case  $N = 1$ .

Below, when speaking of equations in words, we assume that the left-hand and right-hand sides of an equation are not graphically equal.

It is known (see, for example, <sup>(2)</sup>) that coefficient-free equations in words with two unknowns have no nontrivial solutions. Equations with more than two unknowns may also have nontrivial solutions.

For example, the equation  $x_1 = x_2 x_3$  has a nontrivial solution ( $X_1 = a_1 a_2$ ,  $X_2 = a_1$ ,  $X_3 = a_2$ ).

In the present note we formulate Theorem 2, which establishes the structure of nontrivial solutions of coefficient-free equations in words with three unknowns. From this theorem, in particular, there follows an algorithm which, for a given coefficient-free equation in words with three unknowns, recognizes whether this equation has nontrivial solutions.

We proceed to the formulation of Theorem 2.

We begin with the definition of parametric words. Let us define variables of two sorts: a)  $W$ -variables, whose values are words in the alphabet . We shall need only two  $W$ -variables, which we denote by the letters  $u, v$ .

- b) Natural variables, whose values are natural numbers. We shall assume that these variables form the alphabet  $\{\lambda_1, \dots, \lambda_m\}$  and that the alphabets  $\{\lambda_1, \dots, \lambda_m\}, \{u, v\}$  have no letters in common. The letters  $u, v, \lambda_1, \dots, \lambda_m$  will be called variables.

We now define parametric words. The inductive definition of parametric words has the following form: 1) if  $T$  is a word in the alphabet  $\{u, v\}$ , then  $T$  is a parametric word; 2) if  $T$  is a parametric word and  $\lambda$  is a natural variable, then  $(T)^\lambda$  is a parametric word; 3) if  $T_1, T_2$  are parametric words, then  $T_1 T_2$  is a parametric word; 4) there are no other parametric words except those defined according to 1)–3).

We now define the result of the substitution

$$S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} T \Big|$$

where  $T$  is a parametric word;  $U, V$  are words in the alphabet ;  $a_1, \dots, a_m$  are natural numbers. 1) if  $T$  is a word in the alphabet  $\{u, v\}$ , then

$$S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} T \Big|$$

is

$$S_{U,V}^{u,v} T \Big|;$$

2) if  $T$  is a parametric word,  $\lambda_i$  is a natural variable ( $1 \leq i \leq m$ ), then

$$S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} (T)^{\lambda_i} \Big|$$

is

$$\left( S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} T \Big| \right)^{\lambda_i};$$

3) if  $T_1, T_2$  are parametric words, then

$$S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} T_1 T_2 \Big|$$

is

$$S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} T_1 \Big| S_{U,V,a_1,\dots,a_m}^{u,v,\lambda_1,\dots,\lambda_m} T_2 \Big|.$$

Let  $\xi_1, \dots, \xi_q$  be a list of variables;  $P(\xi_1, \dots, \xi_q)$  a predicate. A set  $(\bar{\xi}_1, \dots, \bar{\xi}_q)$  of values of the variables  $\xi_1, \dots, \xi_q$  will be called **admissible** for the predicate  $P$  if  $P(\bar{\xi}_1, \dots, \bar{\xi}_q)$  is true. The predicate  $P(\xi_1, \dots, \xi_q)$  will be called **nonempty** if there exists at least one set of values of the variables  $\xi_1, \dots, \xi_q$  admissible for the predicate  $P$ .

We now formulate Theorem 2.

**Theorem 1.** Let  $\Phi = \Psi$  be a coefficient-free equation in words with three unknowns  $x_1, x_2, x_3$ . Then one can construct a system of  $L$  sets ( $L > 0$ )

- 1)  $T_{11}, T_{12}, T_{13}, P_1(u, v, \lambda_1, \dots, \lambda_m),$   
 $\dots \dots \dots \dots \dots \dots$
- L)  $T_{L1}, T_{L2}, T_{L3}, P_L(u, v, \lambda_1, \dots, \lambda_m),$

where  $T_{ij}$  are parametric words,  $P_i$  are nonempty decidable predicates, such that, if the equation  $\Phi = \Psi$  has a nontrivial solution, then:

- a) whatever the nontrivial solution  $(X_1, X_2, X_3)$  of the equation  $\Phi = \Psi$ , there exist an integer  $1 \leq \nu \leq L$ , words  $U, V$ , and natural numbers  $a_1, \dots, a_m$  such that  $P_\nu(U, V, a_1, \dots, a_m)$  is true and

$$X_i = S_{U, V, a_1, \dots, a_m}^{u, v, \lambda_1, \dots, \lambda_m} T_{\nu i} \quad (i = 1, 2, 3). \quad (1)$$

- b) Whatever the integer  $1 \leq \nu \leq L$ , the words  $U, V$  in the alphabet  $I$ , and the natural numbers  $a_1, \dots, a_m$  such that  $P_\nu(U, V, a_1, \dots, a_m)$  is true, the system of words (1) is a nontrivial solution of the equation  $\Phi = \Psi$ .

Thus, the nontrivial solutions of coefficient-free equations with three unknowns are representable by means of parametric words.

Such a situation no longer holds for equations with more than three unknowns. One can show, for example, that the totality of nontrivial solutions of the equation  $x_1 x_2 x_3 = x_3 x_4 x_1$  is not representable (in the corresponding precise sense) by means of parametric words. However, this totality can be represented by using the term  $G$  from (1).

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*

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